ELEMENTS

OF

GEOMETRY;

WITH THEIR

Application to the Menſuration of SUPERFICIES and SOLIDS,

TO THE

Determination of the MAXIMA and MINIMA of Geometrical Quantities,

AND TO THE

Conſtruction of a great Variety of GEOMETRICAL PROBLEMS.

By *THOMAS SIMPSON*, F. R. S.

And Member of the Royal Academy of Sciences at STOCKHOLM.

The FOURTH EDITION,

Carefully Reviſed.

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MDCCLXXX.

TO THE HONOURABLE

Charles Frederick, Esq;

Surveyor-General of his MAJESTY'S ORDNANCE, &c. &c. &c.

HONOURABLE SIR,

THE subject of the sheets which I here beg leave to lay before you, is of so much consequence to mankind, as justly to claim the regard and sanction of the Great. Geometry is, not only a most accurate, but a very extensive science, whose application and great utility, as well in the arts of peace as of war, are well known to You.

But though this work, if the manner in which it is executed be correspondent to the importance of the subject, may not want sufficient merit to render it worthy of the approbation of a Gentleman, who, amidst a multiplicity of public employments, preserves an undiminish'd ardor for the sciences,

and a knowledge of the works of art and nature; yet I have, Sir, ſtill farther motives for this addreſs: Your great influence and zeal to promote the good of an inſtitution under which I am placed; and the favours that I have received at your hands, make me earneſt to embrace this opportunity of teſtifying publickly, that I am,

HONOURABLE SIR,

With great reſpect,

Your much obliged,

and moſt obedient

humble ſervant,

Royal Academy,
March 3, 1760.

Thomas Simpſon.

PREFACE.

MY design in writing upon the subject of Geometry, was to open an easy way for young beginners to arrive at a proficiency in that useful science; without either being obliged to go thro' a number of unnecessary propositions, or having recourse to the ungeometrical *methods of demonstration, that abound in most modern compositions of this nature.*

The difficulty of the undertaking, I was not unapprised of; and objections occurred that were not easy to be removed: Neverthelefs, I have grounds to hope, from the reception my first attempt has met with, that my endeavours have not been entirely unsuccessful. No pains have, indeed, been spared to render the work useful: And I flatter myself, that the spirit and rigour of demonstration, so essential to the subject, are also tolerably well preserved; though I have not been so intent to guard against the attack of Criticks, *as to lose sight of my main design of furnishing a plain, easy institution for learners: Yet I have strong hopes, that there will not be found in these sheets, any inaccuracies, or oversights, that are absolutely unpardonable. To expect a faultless piece is impossible: And I well know that the most elaborate and best-approved systems of Geometry extant, are not without many imperfections. But, were the smallest imperfection to be a real fault, my ambition would rather be, to shew some degree of judgment, by avoiding a multitude of such faults, than by exposing and magnifying the flaws of other writers. It is more easy to see a fault, than to avoid one: And those men who are the most sanguine to distinguish themselves at the expence of others, are*

generally observed to stand in need of greater indulgencies, than even the persons whom they unmercifully attack. But I shall put an end to this digression by pointing out one objection, that may be brought against this work; which is, that in demonstrations admitting of several cases, the most easy ones are sometimes omitted; and that the converse of some propositions is not at all demonstrated. But this, I conceive, will be found a real advantage to the learner; without which, it would have been impossible to have comprised the Elements in the compass they now take up. Besides, the greatest part of the demonstrations omitted being such as may be inferred from those given, by means of Axioms only; they may, therefore, be easily supplied by any reader, should they happen to become necessary, which I have scarce ever found to be the case. But, even allowing this to be a defect, it is abundantly compensated by the extensive application given in the three last sections; which is infinitely more useful, in itself, and more necessary to the forming an able Geometrician, *than any thing of the kind we have been speaking of.*

In this, second, edition (which is, in a manner, a new work) many considerable alterations and additions have been made. The order of some of the first propositions is changed: And some difficult propositions in the second book are rendered more plain. In the fourth book several new Theorems on proportions are added. The solid Geometry is now connected with the plane, *and is demonstrated with the same accuracy. The mensuration of Superficies and Solids is also more explicitly handled; and the demonstration of the several rules is here established on a better foundation, than even in authors who have wrote professedly on the subject. The Maxima* and *Minima, and the construction of Geometrical Problems, are likewise considerably extended and improved. And, at the end,*

Notes

Notes geometrical and critical, very useful to improve the judgment of young students, are now added.

But, whilst I am talking of improvements and matters of criticism, I am called upon to answer to a charge, which, should it appear to deserve credit, would indeed leave me but little room to pass myself upon the world for a judge in these matters. As the gentleman by whom I stand accused, is known to the world by his holding one of the most considerable mathematical posts in the kingdom; I shall, in order to do all due honour to the manner and importance of his writing, give you his own words.

" *There has lately been published a book under the*
" *title of Elements of Plane Geometry, designed for*
" *the use of schools, which is an incorrect copy of the*
" *first eight sections of this work, lent the pretended*
" *author on a particular occasion, and printed in a*
" *spurious manner, without my knowledge or consent;*
" *an action too scandalous for any man of honour to*
" *be guilty of. The Editor imagined, I suppose, that*
" *the changing some propositions, and mangling the*
" *demonstrations of others, was a sufficient disguise*
" *to make it pass for his own performance; but how*
" *far this will justify such a piece of piracy, must be*
" *left to the judgment of the publick.*"

Were I to attempt to describe the ideas excited in my mind by the singular modesty *of this important and solemn appeal to the publick, I should be at a loss for fit words to express them, without transgressing the bounds of decency. But I hope that I have not deserved so ill of the publick, to be thought capable of acting so very humble a part, as that of copying from* this *author, and of mangling* his *demonstrations, in order to make them pass for* my own.—*That a manuscript of his (containing between* 20 *and* 30 *of the principal*

principal Theorems in Geometry, extremely ill digested) came into my hands, is indeed true; but it was not lent me, *but forced upon me, by himself (the very first night after my removal to* Woolwich) *in virtue of an* article *in the original rules and instructions for the Academy; whereby it is ordered, that the second master shall teach Geometry under the direction of the first master. But this well intended* article, *which has been made subservient to the purposes of ignorant tyranny, and daring calumny, has since, in consequence of a publick examination, been annulled by an express order of the Master-General of the Ordnance.—I could mention some particulars, supported by good authority, that occurred in the course of that examination, which would but ill agree with the importance he assumes in his confident accusation; but I do not think it worth while: This Gentleman has, himself, by his different publications, so well convinced the world of his abilities, as to render any farther comment on that head intirely unnecessary and ineffectual.*

ADVER-

ADVERTISEMENT.

AS in every work of this nature, defigned to contain whatever may be moft requifite to the forming of a regular and complete fyftem of Geometry, a number of propofitions muft neceffarily have a place, whofe chief ufe and application lie in the higher branches of the Mathematics; and there being many perfons, particularly young gentlemen in publick fchools, who want to learn fo much Geometry only, as is neceffary to give them a proper introduction into the practical and moft common applications thereof; fuch as Menfuration, Trigonometry, Navigation, Fortification, Perfpective, *&c.* For thefe reafons, I thought that it might be of fervice, to point out to fuch Readers, what propofitions in thefe elements may be omitted, as leaft ufeful to them; without either hurting the connection, or taking away from the evidence of the other demonftrations. The numbers of thefe propofitions, in the feveral books, are as follow.

In Book I. the 6, 17, 19, 21, 22, 23, and 29th.

In Book II. the 4, 5, 10, 11, 12, 13th, and the 2d Corol. to the 9th.

In Book III. the 4, 5, 6, 7, 8, 9, 15, 18, 19, 20, 25, 26, 27, and 28th.

In Book IV. the 4, 5, 6, 9, 11, 13, 16, 17, 20, 21, 22, 23, 25, 26, 27, 28, and 29th.

In

In Book V. the 1, 2, 16, 17, 18, 19, 20, 25, 26, 28, and 31ſt.

In Book VI. the two or three firſt propoſitions only, need be read; except by thoſe who are concerned in ſurveying and dividing of lands; to whom the whole Book will be highly uſeful.

Alſo, with regard to the ſeventh book, if *Perſpective* be the only application in view (which I have known frequently to be the caſe) the 1ſt, 2d, 4th, and 12th propoſitions may ſuffice. But if a more general idea of the properties of interſecting planes ſhould be required, ſuch as is neceſſary in the doctrine of ſolids and ſpheric geometry; then all the propoſitions, to the 12th, ought to be taken.

The 17th, 19th, 20th, 21ſt, 22d, and 23d propoſitions of this ſeventh Book ſhould alſo be read by thoſe who would be able to find the content and proportion of ſolid bodies; as ſhould, likewiſe, the whole eighth book; except, perhaps, the firſt and ninth propoſitions, together with the three firſt lemmas; which may be thought too plain, by thoſe who are not very ſolicitous about geometrical rigour, to need a demonſtration.

An INDEX *or* TABLE *referring to the places in these Elements, where all the most material propositions in the first six, and in the eleventh and twelfth books of* Euclid, *are demonstrated.*

Euclid.	These El.
Book I.	P. B.
Prop. 5	12. 1
6	18. 1
8	14. 1
9	5. 5
10	6. 5
11	3. 5
12	4. 5
13	1. 1
15	3. 1
18, 19	13. 1
21	23. 1
22	8. 5
23	7. 5
24	21. 1
26	15. 1
27, 28	8. 1
29	7. 1
31	9. 5
32	9. 1, 10. 1
33	26. 1
34	24. 1
35	2. 2
36, 37, 38, 41	Cor. to 2. 2
42	6. 6
43	3. 2
44	6. 6
45	7. 6
46	10. 5
47	8. 2
B. II.	P. B.
Pr. 1.	5. 2
4	6. 2
5	7. 2

Euclid.	These El.
B. II.	P. B.
P. 11	19. 5
12, 13	10. 2
14	9. 6
B. III.	P. B.
Pr. 3	2. 3
7, 8	5. 3
11	8. 3
12	7. 3
14	3. 3
15	4. 3
17	21. 5
20	10. 3
21	11. 3
22	17. 3
25	18. 3
31	16. 3
32	14. 3
33	22. 5
34	25. 5
35	21. 3
36, 37	22. 3
B. IV.	P. B.
Pr. 2	25. 5
3	26. 5
4	24. 5
5	23. 5
10, 11	28. 5
12	30. 5
15	29. 5
B. V.	P. B.
4	4. 4
12	6. 4
15	1. 4

Euclid.	These El.
B. V.	P. B.
P. 16	2. 4
17, 18, 19	3. 4
22	5. 4
24	6. 4
25	3. 4
B. VI.	P. B.
Pr. 1	7. 4
2	12. 4
3	18. 4
4	14. 4
5	17. 4
6	15. 4
7	16. 4
8	19. 4
9	11. 5
10	15. 5
11	12. 5
12	13. 5
13	14. 5
14, 15	Cor. to 25. 4
16, 17	10. 4
18	11. 6
19	24. 4
20	26. 4
22	28. 4
23	25. 4
25	13. 6
27, 28	17. 5
29	18. 5
30	19. 5
31	29. 4
B. XI.	P. B.
Pr. 3	1. 7
4	2. 7

Euclid.	These El.
B. XI.	P. B.
Pr. 5	Cor. to 2. 7
6	4. 7
8	5. 7
9	8. 7
10	9. 7
12	3. 7
14	7. 7
15	10. 7
16	11. 7
18	6. 7
19	13. 7
24	16. 7
25	21. 7
28, 29, 30, 31	17, 19 and 20 of the 7th, and Cor. 3. to 8. 8.
32	21. 7
33	9. 8
34	23. 7
35	14. 7
37	14. 7, 25. 7
B. XII.	P. B.
Pr. 2	3. 8
5, 6	Cor. 4 to 8. 8
7	7. 8
8	9. 8
9, 10, 11, 12, 13, 14, 15	Cor. 3, 4 and 6. to 8. 8.
18	Cor. to 11. 8

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ELEMENTS

OF

GEOMETRY.

BOOK I.

DEFINITIONS.

1. GEOMETRY is that ſcience, by which we compare ſuch quantities together as have extenſion.

Extenſion is diſtinguiſhed into length, breadth, and thickneſs.

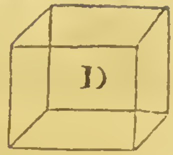
2. A Line is that, which has length without breadth.

The terms, bounds or extremes of a Line are points.

3. A Surface is that, which has length and breadth, only, as C.

The bounds of a Surface are lines.

4. A Solid is that, which has length, breadth, and thickneſs, as D.



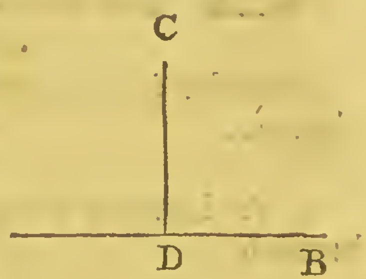
The bounds of a Solid are ſurfaces.

5. A Right (or ſtrait) line is that which lies evenly between its extremes, or which every-where tends the ſame way, as AB. A——B

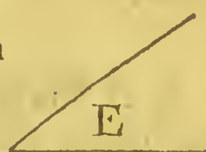
6. A Plane ſurface is that, which is every-where perfectly flat and even, or which touches, in every part, any right line extended between points any-where taken in that ſurface.

7. An Angle is the inclination, or opening of two right-lines meeting in a point, as D.

8. When one right-line DC, ſtanding upon another AB, makes the angles on both ſides equal, thoſe angles are called right angles; and that line CD is ſaid to be perpendicular to the other AB on which it inſiſts.



9. An Acute-angle is that, which is leſs than a right-angle, as E.



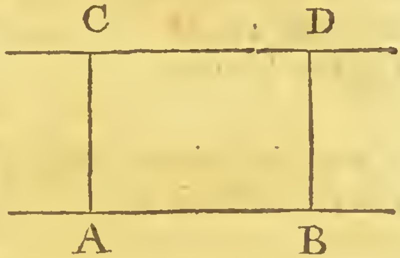
10. An Obtuſe-angle is that, which is greater than a right-angle, as F.

11. The

11. The diſtance of two points, is the Right-line reaching from the one to the other.

12. The diſtance of a point from a line, is a Right-line drawn from that point, perpendicular to, and terminating in, the line given.

13. Parallel (or equidiſtant) right-lines AB, CD are ſuch, which being in the ſame plane-ſurface, if infinitely produced, would never meet.



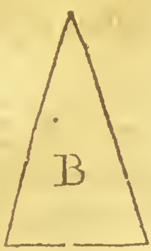
14. A Figure is a bounded ſpace, and is either a ſurface, or a ſolid.

15. A right-lined plane Figure is that, formed in a plane ſurface, whoſe terms, or bounds, are right-lines.

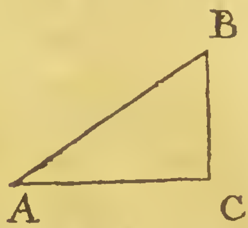
16. All plane Figures bounded by three right-lines, are called Triangles.

17. An equilateral Triangle is that, whoſe bounds or ſides are all equal, as A.

18. An iſoſceles Triangle is, when two ſides are equal, as B.



19. A ſcalene Triangle is, when all the three ſides are unequal, as C.



20. A right-angled Triangle is that, which has one right-angle, as ACB; whereof the ſide AB oppoſite to the right-angle, is called the Hypothenuſe.

21. An obtuſe-angled Triangle is that, which has one obtuſe angle.

22. An acute-angled Triangle is that, which has all its angles acute.

23. Every plane Figure bounded by four right-lines, is called a Quadrangle, or Quadrilateral.

24. Any Quadrangle, whoſe oppoſite ſides are parallel, is called a Parallelogram, as D.

25. A Parallelogram, whoſe angles are all right-ones, is called a Rectangle, as E.

26. A Square is a parallelogram whoſe ſides are all equal, and its angles all right-ones, as F.

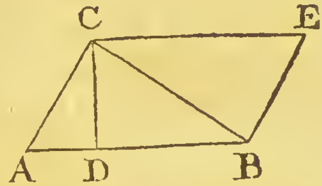
27. A Rhombus is a parallelogram whoſe ſides are all equal, but its angles not right, as G.

28. All

28. All other four-ſided figures, beſides theſe, are called trapeziums.

29 A right line joining any two oppoſite angles of a four-ſided figure, is called a Diagonal.

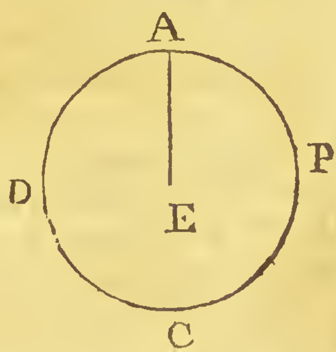
30. That ſide AB upon which any parallelogram ACEB, or triangle ACB is ſuppoſed to ſtand, is called the baſe; and the perpendicular CD falling thereon from the oppoſite angle C, is called the altitude of the parallelogram, or triangle.



31. All plane figures contained under more than four ſides, are called polygons; whereof thoſe having five ſides, are called Pentagons; thoſe having ſix ſides, Hexagons; and ſo on.

32. A Regular Polygon is one whoſe angles, as well as ſides, are all equal.

33. A Circle is a plane figure, bounded by one curve-line APCD, called its circumference, every-where equally diſtant from a point E within the circle, called the center thereof.



34. The Radius of a circle, is the diſtance of the center from the circumference, or a right-line EA drawn from the center to the circumference.

AXIOMS, *or* Self-evident Truths.

1. Things, equal to one and the ſame thing, are alſo equal to each other.

2. Every whole is greater than its part.

3. Every whole is equal to all its parts taken together.

4. If to equal things, equal things be added, the wholes will be equal.

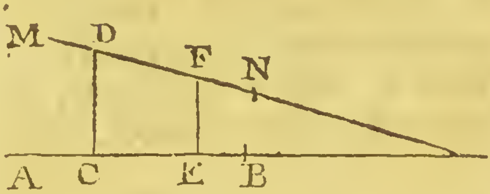
5. If from equal things, equal things be taken away, the remainders will be equal.

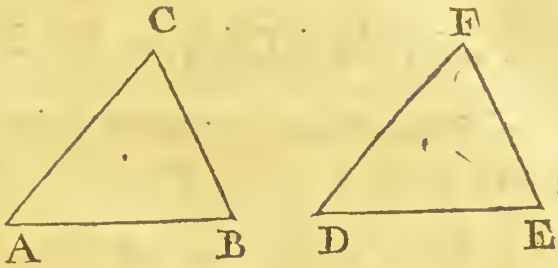
6. If to, or from unequal things, equal things be added, or taken away, the ſums, or remainders, will have the ſame difference, as the unequal things firſt propoſed.

7. All right-angles are equal to one another.

8. More than one right-line cannot be drawn from one given point A to another given point B.

9. If two points D, F, in a right-line MN, are poſited at unequal diſtances DC, FE, from another right-line AB in the ſame plane-ſurface; thoſe two lines, being infinitely produced, on the ſide of the leaſt diſtance EF, will meet each other.





10. If two right-lines CA, CB, making an angle C, be reſpectively equal to two other right-lines FD, FE, making an angle F, and the angles which they make C, and F, be likewiſe equal; the right-lines AB, DE joining their extremes will be equal, and the two triangles ACB, DFE equal in all reſpects.

If this ſhould not appear ſufficiently evident for an axiom; conceive the triangle DFE to be removed, and ſo applied to the triangle ABC, that the point F may coincide with C, and the ſide FD fall upon the ſide CA; then, becauſe FD is ſuppoſed equal to CA, the point D will alſo fall upon A. And, the angle F being equal to the angle C, the ſide FE will fall upon CB; and conſequently the point E upon the point B, becauſe FE is ſuppoſed equal to CB. Therefore, ſeeing all the bounds of the two triangles coincide, it is manifeſt, that not only the baſes AB, DE, but the angles oppoſite to the equal ſides, are alſo equal.

When all the four lines CA, CB, FD, FE are equal; the triangle DFE, being *contrariwiſe* applied to ACD ſo that FE may coincide with CA, will, *alſo*, agree with the triangle ACB (as is manifeſt from the reaſoning above): and ſo, the angle E (as D did before) now coinciding with the angle A, the two angles E and D muſt neceſſarily be equal to each other, in this caſe, where the triangle DFE is an iſoſceles one.

POSTULATES, *or* PETITIONS.

1. That, from any given point, to any other given point, a right-line may be drawn.

2. That, a right line may be produced, or continued out, at pleaſure.

3. That, from any point as a center, with a radius equal to any right-line aſſigned, a circle may be deſcribed.

4. That, a right-line may be drawn perpendicular to another, at any point aſſigned; and that it is alſo poſſible for to make a right-line, or a right-lined angle, equal to any right-line, or right-lined angle aſſigned, or to the half thereof.

This fourth Poſtulate *is added, more for the ſake of making the proper references, than through abſolute neceſſity: ſince, what is here barely aſſumed as poſſible is effected, and actually demonſtrated, in the beginning of the Fifth Book, intirely independent of every thing but* Axioms *and the other Poſtulates, above laid down. It may alſo be proper to note here, that, though theſe* Poſtulates *are not always quoted, it will be eaſy to perceive where, and in what ſenſe, they are to be underſtood.*

NOTES *and* OBSERVATIONS, *with the ſignifications of Signs uſed in this Tract.*

A PROPOSITION is, when ſomething is, either, propoſed to be done, or to be demonſtrated, and is either a Problem, or a Theorem.

A PROBLEM is, when ſomething is propoſed to be done.

A THEO-

A Theorem is, when something is proposed to be demonstrated.

A Lemma is, when some premise is demonstrated, in order to render the thing in hand the more easy.

A Corollary is, a consequent truth, gained from some preceding truth, or demonstration.

A Scholium is, when remarks and observations are made upon something going before.

The signification of Signs.

The sign $=$, denotes that the quantities betwixt which it stands, are equal.

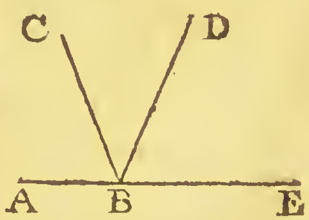
The sign $\sqsubset$, denotes that the quantity preceding it, is greater than that which comes after it.

The sign $\sqsupset$, denotes that the quantity preceding it, is less than that which comes after it.

The sign $+$, denotes that the quantity which it precedes, is to be added.

The sign $-$, denotes that the quantity which it precedes, is to be taken away or subtracted.

A figure, or number, prefixed to any quantity, shews how often that quantity is to be taken, or repeated; as 5A shews, that the quantity represented by A, is to be taken 5 times.



When several angles are formed about the same point (as at B), each particular angle is described by three letters, whereof the middle one shews the angular point, and the other two, the lines that form the angle: thus CBD or DBC signifies the angle formed by the lines CB and DB.

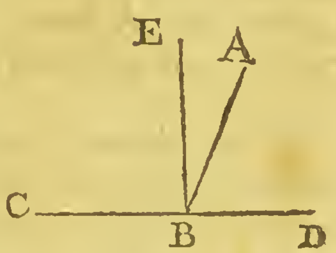
When,

When, in any demonſtration, you meet with ſeveral quantities joined the one to the other continually by the mark of equality (=), the concluſion drawn from thence, is always gathered from the firſt and laſt of them; which are equal to each other, by virtue of the firſt axiom. Thus if A=B=C=D, then will the firſt (A) and the laſt (D) be equal to each other.

Alſo, when in the quotations you meet with two numbers, the firſt ſhews the propoſition, and the ſecond the book. Moreover, Ax. denotes axiom; Poſt. poſtulatum; Def. definition; Hyp. hypotheſis. Note alſo, that, when-ever the word *Line* occurs, without the addition of either *right*, or *curved*, a right-line is always underſtood: and that, when a line is ſaid to be drawn to, or from an angle, the angular point is meant.

THEOREM I.

A line (AB) *ſtanding upon another line* (CD) *makes with it two angles* (ABC), (ABD) *which, taken together, are equal to two right angles.*



If the angles ABC, ABD are equal, it is plain they
[a] Def. 8. make two right-angles[a]; if unequal, let BE be perpen-
[b] Poſt. 4. dicular to CD[b], dividing the greater of them (ABC) into the parts EBC, EBA; then the former part EBC being a right-angle[a], and the remaining part EBA together with the whole leſſer angle ABD, equal to another right-
[c] Ax 3. angle EBD[c]; the whole, of both the propoſed angles, taken together, muſt neceſſarily be equal
[d] Ax. 4. to two right-angles[d].

COROL-

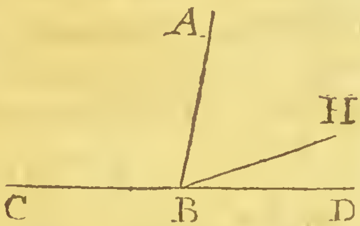
COROLLARY.

Hence all the angles at the same point (B) on the same side of a right line (CD) are equal to two right-angles [c]. [c] Ax. 3.

THEOREM II.

If one line (AB) *meeting two others* (BC, BD) *in the same point* (B), *makes two angles with them* (ABC, ABD) *which together are equal to two right angles; these lines* (BC, BD) *will form one continued right-line.*

For, *if possible*, let BH, and not BD, be the continuation of the right-line CB: then the angles ABC and ABH being = two right angles [e] = ABC and ABD [f]; if from these equal quantities, ABC, common to both, be taken away, there will remain ABH = ABD [g]; *which is impossible* [h].

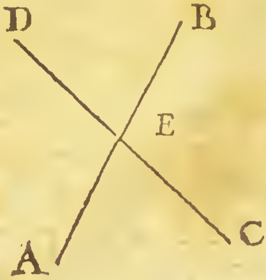


[e] 1. 1.
[f] Hyp.
[g] Ax. 5.
[h] Ax. 2.

THEOREM III.

The opposite angles (DEB, AEC), *made by two lines* (DC, BA) *intersecting each other, are equal.*

For DEB + DEA = two right-angles [i] = AEC + DEA; whence, by taking away DEA, common, there remains DEB = AEC [k].



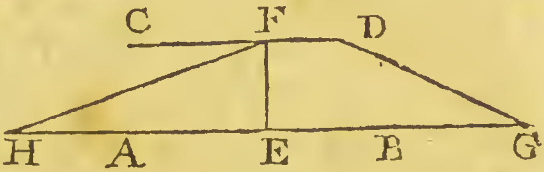
[i] 1. 1.
[k] Ax. 5.

THEO-

THEOREM IV.

Two right-lines (AB, CD) *perpendicular to one and the same right-line* (EF), *are parallel to each other.*

If you say, they are not parallel; then let them, when produced out, meet in some point, as G.



In EA, produced (if necessary) let there be taken EH =
l Post. EG [l], and let
m Post. 1. the right-line FH be drawn [m]. The triangles EHF and EGF, having EH = EG, the angle HEF =
n Def. 8. GEF [n], and EF common, are therefore equal in all
o Ax. 10. respects [o]: and so, the angle EFH being = EFG
p Hyp. (EFD) = a right-angle [p], HFDG (as well as HEG)
q 2. 1. must be one continued right-line [q]: *which is im-*
r Ax. 8. *possible* [r]. Therefore AB and CD are parallels.

SCHOLIUM.

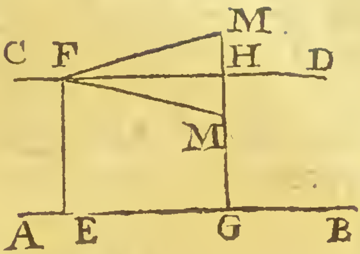
In this theorem, the possibility of parallel lines (or such, which being infinitely produced, in the same plane, can never meet) is demonstrated: for EF may be drawn perpendicular to AB [l]; and CFD, again, perpendicular to EF [l]; which last, it is demonstrated, will be parallel to AB.

THEOREM V.

Perpendiculars (EF, GH) *to one* (AB) *of two parallel lines* (AB, CD) *terminated by those lines, are equal to each other; and also perpendicular to the other of the two parallels* (CD).

For, AB and CD being parallel to each other,
s Ax. 9. GH can neither be greater, nor less than EF [s];
and Def. and therefore must be equal to EF. If you say,
13. that EF is not perpendicular to CD; then let FM
t Post. 4. be perpendicular to EF [t], meeting GH produced (if ne-

neceſſary) in M: ſo ſhall FM be parallel to AB[u]; and conſequently GM = EF[w] = GH; *which is impoſſible*[x]. Therefore EF is perpendicular to CD. And by the ſame argument, GH is perpendicular to CD.



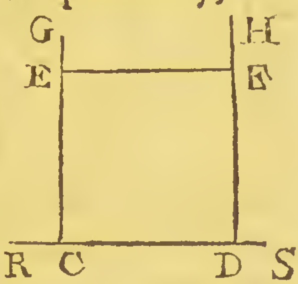
u 4. 1.
w 5. 1.
x Ax. 2.

COROLLARY.

Hence, through the ſame point F, more than one parallel cannot be drawn to the ſame line given AB.

SCHOLIUM.

From the preceding propoſition, the conſiſtence of the twenty-fifth definition, or the poſſibility, that all the properties aſcribed to a rectangle, can ſubſiſt together in the ſame figure, will appear, together with the method of conſtruction. For at any two points C, D in a right line RS, two perpendiculars CG, DH may be erected[y]; and a perpendicular to one of theſe, at any point E, meeting the other in F, may be drawn. The figure CEFD thus conſtructed will be a rectangle: for CE and DF are parallel[z]; as are alſo CD and EF[z]: therefore the angle F (as well as C, D, and E) is a right-angle[a]. If CE be made = CD, then will the rectangle CEDF have all its ſides equal[b]. Which anſwers to the definition of a ſquare.



y Poſt. 4.
z 4. 1.
a 5. 1.
b 5. 1. and Ax. 1.

THEOREM VI.

Right-lines (AB, EF) *parallel to the ſame right-line* (CD) *are parallel to each other.*

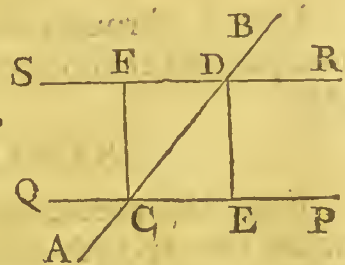
For

For let the line HIG be perpendicular to CD: then, that line being alſo perpendicular to both AB and EF [c], theſe laſt are parallel to each other [d].

[c] 5. 1.
[d] 4. 1.

THEOREM VII.

A line (AB) *interſecting two parallel lines* (SR, QP) *makes the alternate angles* SDC, PCD) *equal to each other.*



Let CF and DE be perpendicular to QP, and SR [e]; then theſe lines FC and DE are likewiſe parallels [f]; and ſo the triangles CFD and CDE, having the ſide CF = DE [c], FD = CE [c], and the angle F = E [g], they will alſo have the angle FDC = ECD [h].

[e] 5. 1.
[f] 4. 1.
[g] Ax. 7.
[h] Ax. 10.

COROLLARY I.

Hence, a line interſecting two parallel lines, makes the angles (BDR, BCP) on the ſame ſide, equal to each other: for BDR (= CDS [i]) = BCP [k].

[i] 3. 1.
[k] 7. 1.

COROLLARY II.

Hence, alſo, a line falling upon two parallel lines, makes the ſum of the two internal angles (SDC + QCD) on the ſame ſide of it, equal to two right-angles: for the angle SDC being = PCD, and PCD + QCD = two right-angles [l]; thence is SDC + QCD = alſo to two right-angles [m].

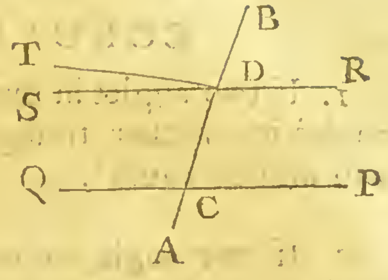
[l] 1. 1
[m] Ax. 4.

THEOREM VIII.

If a line (AB) *interſecting two other lines* (PQ, RS), *makes the alternate angles* (DCP, CDS) *equal to each other; then are thoſe two lines parallel.*

For

For, if poſſible, let ſome other line DT, and not DS, be parallel to PQ [n]; then muſt CDT = DCP [o] = CDS [p]: *which is impoſſible* [q].



[n] Sch. to 4. 1.
[o] 7. 1.
[p] Hyp.
[q] Ax. 2.

COROLLARY.

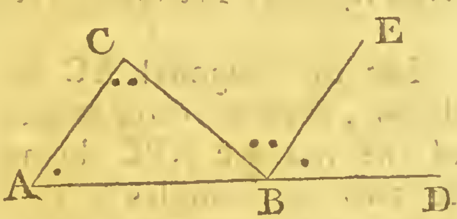
Hence, if a line falling on two others, makes the angles (BDR, BCP) above them, on the ſame ſide, equal to each other; then thoſe two lines are parallels: becauſe SDC = BDR [r].

[r] 3. 1.

THEOREM IX.

If one ſide (AB) *of a triangle* (ABC) *be produced, the external angle* (CBD) *will be equal to both the internal oppoſite angles* (A, C) *taken together.*

For, let BE be parallel to AC [s]; then will the angle C = CBE [t], and the angle A = DBE [u]: therefore C + A = CBE + DBE [x] = CBD [y].



[s] Sch. to 4. 1.
[t] 7. 1.
[u] Cor. to 7. 1.
[x] Ax. 4.
[y] Ax. 3.

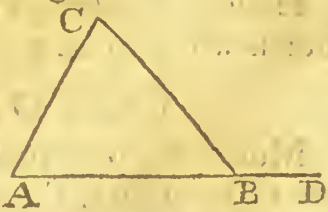
COROLLARY.

Hence the external angle of a triangle is greater than either of the internal, oppoſite angles.

THEOREM X.

The three angles of any plane triangle (ABC) *taken together, are equal to two right-angles.*

For, if AB be produced to D, then C + A = CBD [z], to which equal quantities let the angle CBA be added, then will C + A + CBA = CBD + CBA [a] = two right-angles [b].



[z] 9. 1.
[a] Ax. 4.
[b] 1. 1.

COROL-

COROLLARIES.

1. If two angles in one triangle, be equal to two angles in another triangle, the remaining angles will also be equal [c].

c Ax. 5.

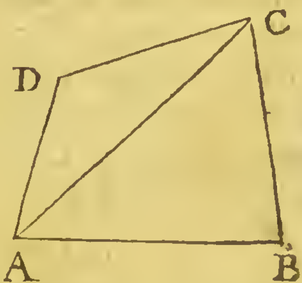
2. If one angle in one triangle, be equal one angle in another, the sums of the remaining angles will be equal [c].

3. If one angle of a triangle be right, the other two taken together, will be equal to a right-angle.

4. The two least angles, of every triangle, are acute.

THEOREM XI.

The four inward angles of a quadrangle (ABCD) *taken together, are equal to four right-angles.*



Let the diagonal AC be drawn; then the three angles of the triangle ABC being = two right-angles [d], and those of the triangle ACD equal also to two right-angles [d]; it follows that the sum of all the angles of both triangles, which make the four angles of the quadrangle, must be equal to four right-angles [c].

d 10. 1.

c Ax. 4.

COROLLARY I.

Hence, if three of the angles be right ones, the fourth will also be a right-angle.

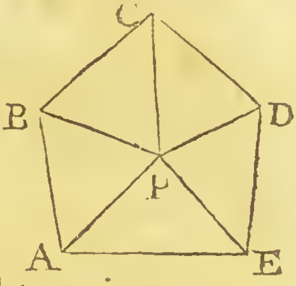
COROLLARY II.

Moreover, if two of the four angles, be equal to two right-angles, the remaining two together will likewise be equal to two right-angles.

SCHO-

SCHOLIUM.

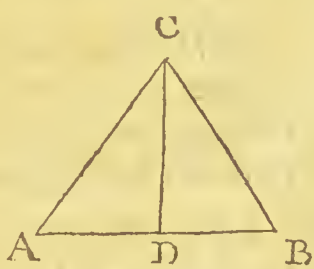
If from any point P, within a polygon ABCDE, lines be drawn to all the angles, ſo as to divide the whole into as many triangles APB, BPC, CPD, DPE, EPA, as the polygon has ſides; the ſum of all the angles of theſe triangles, (which together make up, or compoſe the angles of the polygon, over and above thoſe about the point P) will be equal to twice as many right angles as the polygon has ſides (by 10. 1.) Therefore, ſeeing all the angles about the point P, whereby the angles of all the triangles exceed thoſe of the polygon, are equal to four right angles, it is manifeſt, that all the angles of the polygon, taken together, will be equal to twice as many right-angles, wanting four, as the polygon has ſides.



THEOREM XII.

The angles (A, B,) *at the baſe of an iſoſceles triangle* (ABC) *are equal to each other.*

For, let the line CD biſect, or divide the angle ACB intwo two equal parts ACD, BCD, and meet AB in D: then the triangles ACD, BCD, having AC = BC[f], CD common, and the angle ACD = BCD[g], will alſo have the angle A = B[h].



[f] Def. 18.
[g] Hyp.
[h] Ax. 10.

COROLLARY I.

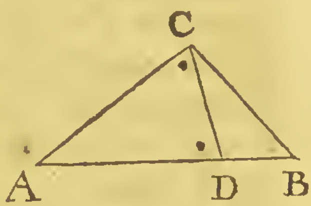
Hence, the line which biſects the vertical angle of an iſoſceles triangle, biſects the baſe, and is alſo perpendicular to it[h].

COROLLARY II.

Hence it appears also, that every equilateral triangle is likewise equiangular.

THEOREM XIII.

In any triangle (ABC) *the greatest side subtends the greatest angle.*



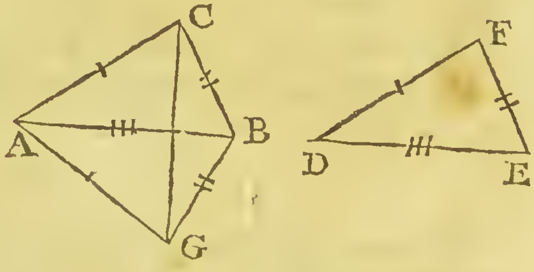
Let AB be greater than AC; in which let there be taken AD = AC; drawing CD. The triangle ADC being isosceles, the angles ACD and
[i] 12. 1. ADC are therefore equal[i]; whence ACB, which exceeds the former of them, must also exceed the
[k] Ax. 2. latter ADC[k], and consequently, much more exceed
[l] Cor. to 9. 1. B, which is less than ADC[l].

COROLLARY.

Hence, in any triangle, the side that subtends the greatest angle, is the greatest; because ACB cannot be greater than B, unless AB is greater than
[m] 13. 1. AC[m].

THEOREM XIV.

If the three sides (AB, AC, CB) *of one triangle, be equal to the three sides* (DE, DF, FE) *of another triangle, each to each respectively; then the angles opposed to the equal sides will also be equal.*



Let the angle BAG = D, AG = DF, and let GB and GC be drawn; so shall the triangles ABG and
[m] Ax. 10. DEF be equal in all respects[m]: therefore, AG
[n] Hyp. being = DF = AC[n], and BG = EF = BC[n], the

the angle ACG is also = AGC [o], and BCG [o] = BGC [o]; and consequently ACB = AGB [p] = [p] DFE: therefore the triangles ABC, DEF are equal in all respects [m].

[o] 12. 1.
[p] Ax. 4 or 5.

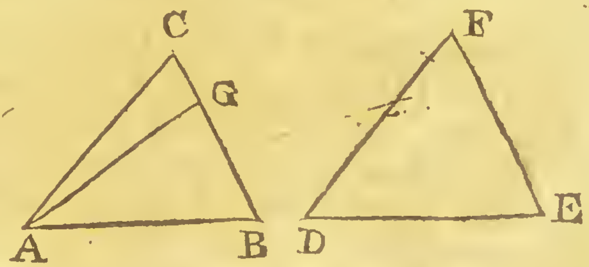
SCHOLIUM.

The demonstration of the last theorem, in obtuse-angled triangles, may admit of another case; which, however, is not necessary; because, if the triangle AGB (equal to DEF) be conceived to be formed on the longest side of ABC; then, all the angles CAB, CBA, GAB, GBA being acute [q], the line CG will, always, fall within the figure ACBG [r], as in the present case.

[q] Cor. 4. to 10. 1.
[r] Ax. 2.

THEOREM XV.

If two triangles (ABC, DEF) *mutually equiangular, have two corresponding sides* (AB, DE) *equal to each other, the other corresponding sides will also be equal.*



If you say BC is greater than EF; from BC let a part BG be taken = EF [s], and let AG be drawn. The triangles ABG, DEF having AB = DE, BG = EF, and B = E *(by hypothesis)*, will also have BAG = D [t]; but D = BAC [u]; therefore BAG = BAC [w]; *which is impossible.*

[s] Post. 4.
[t] Ax. 10.
[u] Hyp.
[w] Ax. 1. and 2.

COROLLARY.

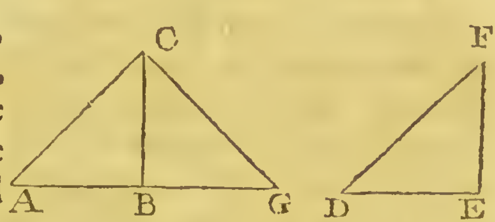
Hence, equiangular triangles, having any two corresponding sides equal, are equal to each other [x].

[x] Ax. 10. 1.

THEOREM XVI.

If two right-angled triangles (ABC, DEF) *having equal hypothenuſes* (AC, DF), *have two other ſides* (BC, EF) *likewiſe equal; the remaining ſides* (AB, DE) *will be equal, and the two triangles equal in all reſpects.*

In AB produced, take BG = ED, and let GC be drawn: then, the triangles BCG and DEF, having BG

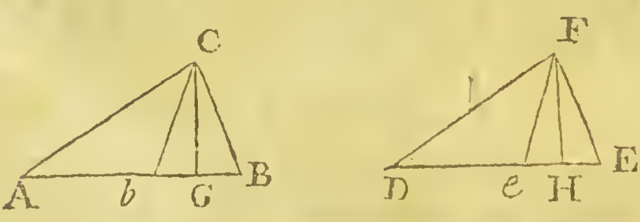


[y] Hyp. [z] Ax. 7. [a] Ax. 10. = ED, BC = EF [y], and the angle CBG = E [z], will alſo have the angle G = D, and CG = DF [a]. = AC [y]: whence, the triangle ACG being iſoſceles, [b] 12. 1. the angle G, or D, will be = A [b]; and conſequently [c] Cor. 1. to 10. 1. F alſo = ACB [c]; therefore the triangles ABC and DEF, being mutually equiangular, and having AC [d] 15. 1. = DF, they are equal in all reſpects [d].

THEOREM XVII.

If two triangles (ABC, DEF) *having two ſides* (AC, BC) *of the one equal to two ſides* (DF, EF) *of the other reſpectively, have alſo the angles* (A, D) *ſubtended by two of the equal ſides* (BC, EF) *equal to each other; and if the angles* (B, E) *ſubtended by the other equal ſides, be either, both acute or both obtuſe; then will the two triangles be equal in all reſpects.*

Let CG and FH be perpendicular to AB and



[e] Ax. 7. DE: then, the angle AGC being = DHF [e], A =

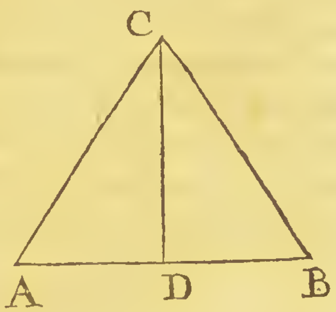
A = D, and the ſide AC=DF [f], CG will alſo be = FH [g]; whence, CB being = FE [f], the angles GBC and HEF are likewiſe equal [h], and ſo, the triangles ABC and DEF, being mutually equiangular [i], and having the ſides AC and DF equal, are equal in all reſpects [g]. [f] Hyp. [g] 15. 1. [h] 16. 1. [i] Cor. 1. to 10.

The demonſtration is the ſame, when both the angles are obtuſe, as in the triangles A*b*C, D*e*F: for, if C*b* (= CB = FE) = F*e*, the angles G*b*C and H*e*F being equal (*as before*), the angles A*b*C and D*e*F will likewiſe be equal [k]. [k] 1. 1. and Ax. 5. 1.

THEOREM XVIII.

If two angles (A, B) *of a triangle* (ABC) *be equal, the ſides* (BC, AC) *ſubtending them will likewiſe be equal.*

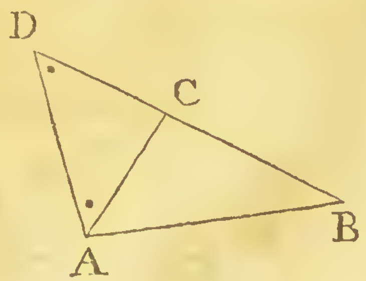
Let CD biſect the angle ACB, and meet AB in D: then the triangles ACD, BCD being equiangular [k], and having CD common to both, they will alſo have AC = BC [l]. [k] Cor. 1. to 10. [l] 15. 1.



THEOREM XIX.

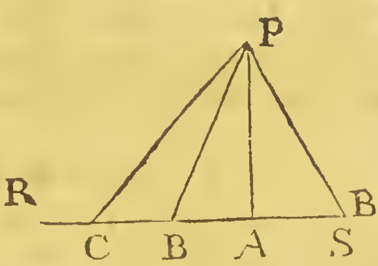
Any two ſides (AC, BC) *of a triangle* (ABC) *taken together, are greater than the third ſide* (AB.)

In BC produced, let there be taken CD = CA, and let AD be drawn. The angles D and DAC are equal [m]; therefore BAD, which exceeds the latter [n], muſt alſo exceed the former D; and conſequently BD (or BC + AC) muſt exceed AB [o]. [m] 12. 1. [n] Ax. 2. [o] Cor. to 13. 1.



THEOREM XX.

Of all the right lines (PA, PB, PC) *falling from a given point* (P) *upon an infinite right line* (RS), *that* (PA) *is the least which is perpendicular to it; and, of the rest, that* (PB) *which is the nearest the perpendicular is less than any other* (PC) *at a greater distance.*



For BAP being a right-angle [p], ABP will be acute [q], and therefore AP ⊐ BP [r].

[p] Hyp. [q] Cor. 4. to 10. 1. [r] Cor. to 13. 1.

Again, when PB and PC are both on the same side of the perpendicular PA; then is CBP ⊏ right angle [s] ⊏ BCP [q], and consequently PC ⊏ PB.

[s] Cor. to 9. 1.

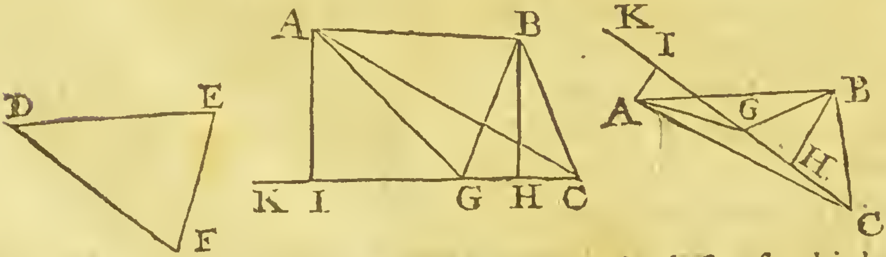
If PB be on the contrary side of the perpendicular to PC; from AC, let AB be taken = AB; then the two lines PB, PB will be also equal [t]; and therefore PC, which exceeds the one (by the preceding case) will also exceed the other.

[t] Ax. 10.

THEOREM XXI.

Of two triangles (ABC, DEF) *having two sides* (AB, BC) *of the one, equal to two sides* (DE, EF) *of the other, each to each respectively, the base of that* (ABC) *will be the greatest, which is subtended under the greatest angle*

Let the angle ABG = E, BG = EF (= BC) also



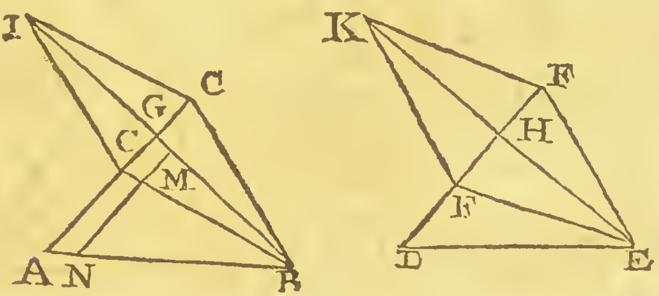
let AG and CG be drawn, upon the last of which, produced,

produced, let fall the perpendiculars BH and AI [u]. [u] 4. 1.
Since BG = BC [w], and, consequently, GH = HC [x], [w] Hyp. [x] 16. 1.
it is evident, that GI (whether the point I be considered as falling between G and K, or between G and H) will be less than CI [z]; and therefore AG, [z] Ax. 2.
or its equal DF [a], also less than AC [b]. [a] Ax. 10. [b] 20. 1.

THEOREM XXII.

Of two triangles (ABC, DEF), *having one angle* (BAC) *in the one equal to one angle* (EDF) *in the other, and the sides* (BC, EF) *opposed to them also equal, that* (ABC) *will have the greatest base, whereof the opposite angle* (ACB) *differs the least from a right-angle.*

Let BG, and EH be perpendicular to AC and DF, in which produced, take HK = HE, GI = GB, and BM = EH; also let MN be parallel to GA, meeting AB, produced if necessary, in N; and let CI and KF be drawn.

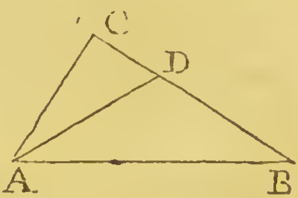


The angle ICG being = BCG [d], and the latter of these greater than EFH [c] (or KFH [d]), thence is [c] Hyp.
ICB $\sqsubset$ KFE; and consequently BI $\sqsubset$ EK [e]; whence [d] Ax. 10.
also BG ($\frac{1}{2}$BI) $\sqsubset$ EH ($\frac{1}{2}$EK) or its equal BM [d]; and [e] 21. 1.
therefore BA $\sqsubset$ BN, because AG and MN being parallels, both the points M and N will fall on the same side of AG. But BN (as the triangles NBM, DEH are equiangular, and have BM = EH [f]) is [f] Hyp. and 7. 1.
= DE [g]: therefore BA is also greater than DE. [g] 15. 1.

THEOREM XXIII.

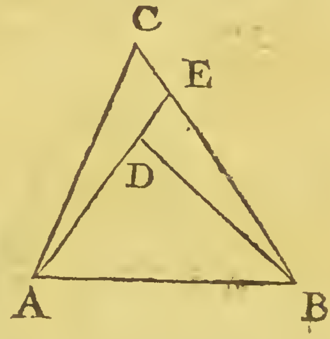
If, of two triangles (ABC, ABD) *ſtanding upon the ſame baſe* (AB), *the one be wholly included within the other, the two ſides* (AD, BD) *of the included one taken together, will be leſs, and the angle* (D) *contained by them greater, reſpectively, than the two ſides* (AC, BC), *and the contained angle* (C) *of the other.*

Case I. *If the vertex of the contained triangle be in one ſide of the other:*



Then, AD is leſs than AC +
[h] 19. 1. CD[h]; whence, by adding BD common, AC + BD will alſo
[i] Ax. 6. be leſs than AC + CD + BD[i],
[k] Ax. 3. or than its equal AC + BC[k].
[l] Cor.9.1. But the angle ADB is $\sqsubset$ ACB[l].

Case II. *If the vertex be within the other triangle.*

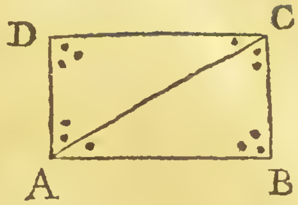


Let AD be produced to meet BC in E: then (*by the preceding caſe*) the ſum of AD and BD is leſs than the ſum of AE and BE; which laſt ſum, and conſequently the former, is, again, leſs than the ſum of AC and BC. Moreover, the angle ADB $\sqsubset$ BED $\sqsubset$ C.

THEOREM XXIV.

The oppoſite ſides (AB, DC) *of any parallelogram* (ABCD) *are equal, as are alſo the oppoſite angles* (B, D); *and the diagonal* (AC) *divides the parallelogram into two equal parts.*

For,



For, AB, DC, and AD, BC being parallels [n], the angle BAC is = DCA [o], and BCA = DAC [o], therefore the equiangular triangles ABC, ADC [p] having AC common, are equal in all refpects [q].

[n] Def. 24.
[o] 7. 1.
[p] Cor. 1. to 10. 1.
[q] 15. 1.

COROLLARY.

Hence, if one angle (B) of a parallelogram be a right-angle, all the other three will be right ones: for D, being = B, is a right angle; and BCD is = B, and DAB = D, *by Theor.* V.

THEOREM XXV.

Every quadrilateral (ABCD) *whofe oppofite fides are equal, is a parallelogram.* (See the preceding fcheme).

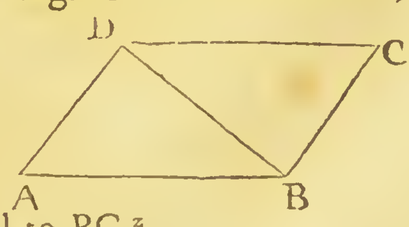
Let the diagonal AC be drawn; then the triangles ABC, ADC being mutually equilateral [r], they will alfo be mutually equiangular [s]; confequently AB will be parallel to DC, and AD to BC [t].

[r] Hyp.
[s] 14. 1.
[t] 8. 1.

THEOREM XXVI.

The lines (AD, BC) *joining the correfponding extremes of two equal, and parallel lines* (AB, DC) *are themfelves equal and parallel.*

Let the diagonal BD be drawn. Becaufe AB and DC are parallel [u], the angle ABD is = CDB [w]; therefore, BA being = DC [x] and BD common, the remaining fides and angles will likewife be refpectively equal [y]; and confequently AD parallel to BC [z].



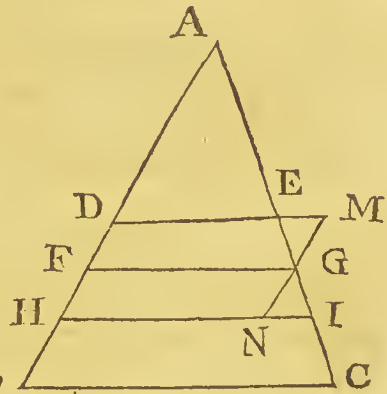
[u] Hyp.
[w] 7. 1.
[x] Hyp.
[y] Ax. 10.
[z] 8. 1.

THEO-

THEOREM XXVII.

If, in one side (AB) *of a triangle* (ABC)*, from three points* (D, F, H) *at equal distances* (DF, FH), *lines* (DEM, FG, HI) *be drawn parallel to the base, the parts* (EG, GI) *of the other side* (AC) *intercepted by them, will also be equal to each other.*

Let NGM be parallel to AB, intersecting HI and DE in N and M. Then, the triangles IGN, MGE, having the
a 3. 1. angle IGN = EGM[a],
b 7. 1. ING = M[b], and GN
c 24. 1. (= FH[c] = FD[d]) =
d Hyp. GM[c], will also have
e 15. 1. GI = GE[e].



COROLLARY I.

Hence it appears, that, if one side of a triangle be divided into any number of equal parts, and from the points of division lines be drawn parallel to the base, cutting the other side, they will also divide it into the same number of equal parts.

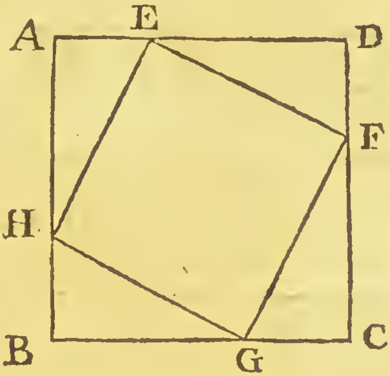
COROLLARY II.

Hence, also, if two lines FG, HI, cutting the sides of a triangle, be parallel to each other, and another line DE be so drawn as to cut off FD = FH and GE = GI, this line DE will be parallel to the two former.

THEO-

THEOREM XXVIII.

If in the ſides of a ſquare (ABCD), *equally diſtant from the four angular points, there be taken four other points* (E, F, G, H,) *the figure* (EFGH) *formed by joining thoſe points, ſhall alſo be a ſquare.*

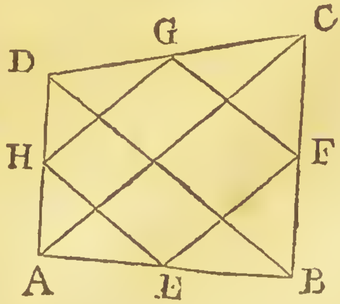


For the wholes AD, DC, CB, BA being equal[f], and alſo the parts AE, DF, CG, BH[g], the remaining parts ED, FC, GB, HA muſt conſequently be equal[h]; whence, all the angles D, C, B, A being equal[i]; the ſides EF, FG, GH, HE will be equal likewiſe[k], and the angle DEF = AHE[k]. Therefore, becauſe DEH is = A + AHE[l], if from theſe, the equal angles DEF, AHE be taken away, there will remain HEF = A[h] = a right-angle[f]. By the ſame argument (or by Theor. 25th, and the Corol. to the 24th) the other three angles will be right-angles.

[f] Def. 26.
[g] Hyp.
[h] Ax. 5.
[i] Ax. 7.
[k] Ax. 10.
[l] 9. 1.

THEOREM XXIX.

If all the ſides of any quadrilateral (ABCD) *be biſected, the figure* (EFGH) *formed by joining the points of biſection, will be a parallelogram.*



Draw the diagonals AC and BD. Becauſe EF and HG are parallel to AC[n], they are alſo parallel to each other[o]. After the ſame manner is FG parallel to EH; therefore EFGH is a parallelogram[p].

[n] Cor. 1. to 27.1.
[o] 6. 1.
[p] Def. 24.

The End of the FIRST BOOK.

ELEMENTS

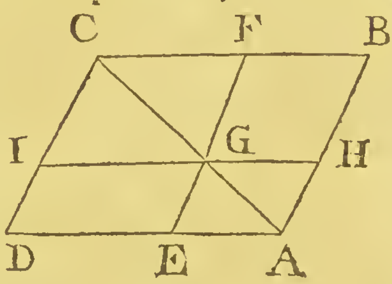
OF

GEOMETRY.

BOOK II.

DEFINITIONS.

1. IN a parallelogram ABCD, if two right-lines EF, HI, parallel to the ſides, interſecting the diagonal in the ſame point G, be drawn, dividing the parallelogram into four other parallelograms; thoſe two GD, GB through which the diagonal does not paſs, are called Complements; and the other two, HE, FI, parallelograms about the diagonal.



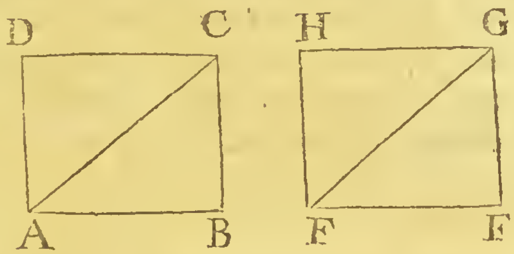
2. Every rectangle is ſaid to be contained under the two right lines AB, BC that are the baſe and altitude thereof.

The

The rectangle contained under two right-lines AB *and* BC *is often, for brevity sake, denoted by* AB×BC. *But when the figure is a square, it is usually represented by placing the number* 2 *over the letter, or letters expressing the side thereof: thus* AB^2 *denotes the square made upon the line* AB.

THEOREM I.

The rectangles (BD, FH) *contained under equal lines, are equal.*

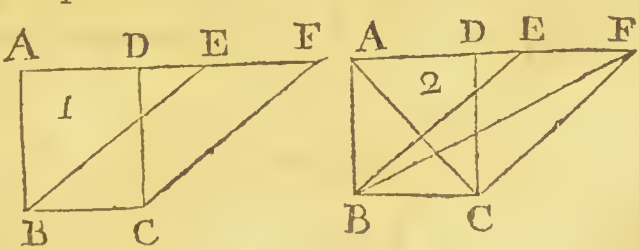


For, let the diagonals AC, EG be drawn: then, because AB = EF, BC = FG, and B = F [a], the triangles ABC, EFG are equal [b]. And, in the very same manner will ADC and EHG appear to be equal. Therefore the whole rectangle ABCD is also equal to the whole rectangle EFGH [c].

[a] Hyp.
[b] Ax. 10.
[c] Ax. 4. 1.

THEOREM II.

Parallelograms (ABCD, BCFE) *standing upon the same base* (BC) *and between the same parallel* (BC, AF) *are equal.*



For, since (in *Fig.* 1.) the angle F = BEA [d], and CDF = A [d], the triangles FDC, EAB are equiangular [e]; they are also equal [f], because CF = BE [g]: therefore, if each be taken from the whole figure ABCF, there will remain ABCD = EBCF [h].

[d] Cor. 1. to 7. 1.
[e] Cor. 1. to 10. 1.
[f] 15. 1.
[g] 24. 1.
[h] Ax. 5.

COROL-

COROLLARY I.

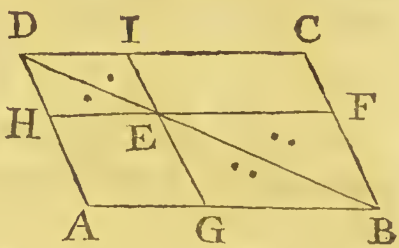
Hence, triangles BAC, BFC (*Fig.* 2.) ſtanding upon the ſame baſe, and between the ſame parallels, are alſo equal, being the halves of their reſpective
[i] 24. 1. parallelograms [i].

COROLLARY II.

Hence all parallelograms, or triangles, whatever, whoſe baſes and altitudes are equal, are equal among themſelves; becauſe all ſuch parallelograms are equal to rectangles ſtanding on the ſame baſes, and between the ſame parallels; and theſe laſt are equal, by the preceding propoſition.

THEOREM III.

The complements (EC, EA) *of any parallelogram* (AC) *are equal.*

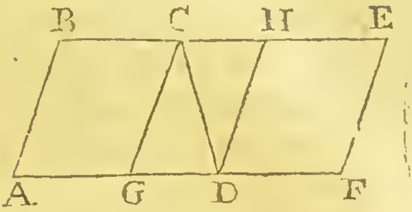


For, the whole triangle DCB being equal to the whole triangle
[k] 24. 1. and Def. 1. DAB [k], and the parts DIE, EFB reſpectively equal to the parts DHE, EGB [k], the re-
[l] Ax. 5. 1. maining parts EC, EA muſt likewiſe be equal [l].

THEOREM IV.

A trapezium (ABCD) *whereof two ſides* (AD, BC) *are parallel, is equal to half a parallelogram, whoſe baſe is the ſum of thoſe two ſides, and its altitude, the perpendicular diſtance between them.*

For,

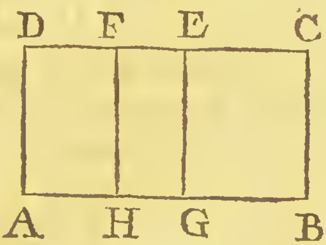


For, in AD produced, take DF = BC; and let CG, DH and FE be all parallel to AB, meeting AF and BC produced, in G, H and E. Then AE is a parallelogram of the ſame altitude with ABCD, having its baſe AF equal to the ſum of AD and BC [n]: but this parallelogram, becauſe BG = HF [o], and CGD = CHD [p], is equally divided by the line CD [q]; and ſo ABCD is the half thereof.

[n] Conſtr.
[o] Cor. 2. to 2. 2.
[p] 14. 1.
[q] Ax. 4.

THEOREM V.

The ſum of all the rectangles contained under a given line (AD), *and all the parts* (AH, HG, GB) *of another* (AB), *any how divided, is equal to the rectangle contained under the two whole lines.*



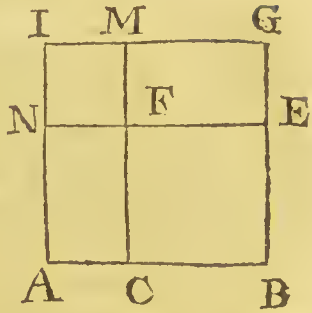
Let ABCD be the rectangle contained under the two whole lines, and let HF, GE be parallel to AD, meeting DC in F and E. Then will AF, HE, GC be rectangles [r] of the ſame altitude with AC [s]; therefore AF = AD × AH, HE = AD × HG, and GC = AD × BG [t]; and conſequently AD × AB (AC = AF + HE + GC) = AD × AH + AD × HG + AD × BG [u].

[r] Cor. to 24. 1.
[s] 24. 1. & Ax. 1.
[t] 1. 2.
[u] Ax. 3. & 4. 1.

THEOREM VI.

If a right-line (AB) *be, any-wiſe, divided into two parts* (AC, BC), *the ſquare of the whole line will be equal to the ſquares of both the parts, together with two rectangles under the ſame parts.*

Let



Let ABGI be the ſquare of AB, and CBEF that of BC, and let EF and CF be produced to meet the ſides of the ſquare ABGI in M and N.

From the equal quantities CM, EN [w] take the equal quantities CF and EF, and there remains FM = FN [x]; therefore, all the angles of the figure being right ones [y], NM is a ſquare [z] upon FN (= AC); and AF, FG are equal to two rectangles under BC and AC [a]: but AG = BF + FI + AF + FG, or $AB^2 = BC^2 + AC^2 + 2AC \times BC$ [b].

[w] 24. 1. and Defin. 26. [x] Ax. 5. 1. [y] Cor. to 24. 1. [z] Def. 26. [a] 1. 2. [b] Ax. 3. 1.

COROLLARY I.

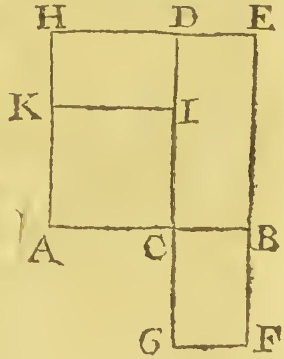
Hence, the ſquare of any line is equal to four times the ſquare of half that line.

COROLLARY II.

Hence, alſo, if two ſquares be equal, their ſides muſt be equal; becauſe unequal lines BA, BC have not equal ſquares.

THEOREM VII.

The difference of the ſquares (ABEH, ACIK) *of any two unequal lines* (AB, AC), *is equal to a rectangle under the ſum and difference of the ſame lines.*



In EB, produced, take BF = AC; let FG be drawn parallel to EH, and let CI be produced both ways, to meet EH and FG in D and G. It is evident that DF is a rectangle [c], whoſe baſe GF (= CB [d]) = the difference of the given lines AB, AC; and whoſe altitude FE (becauſe BE

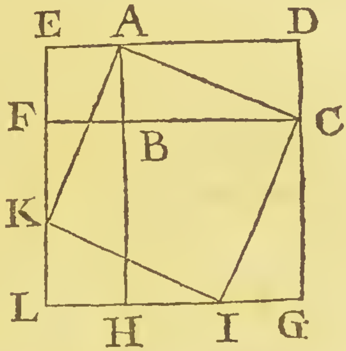
[c] Cor. 24. [d] Ax. 24. 1.

= BA [e], and BF=AC [f]) is=the ſum of the ſame lines: but this rectangle DF is = DB + GB [g] = DB+DK (becauſe DK [h] = GB) = the ſquare AE — the ſquare AI.

[e] Def. 24. [f] Hyp. [g] Ax. 3. [h] 1. 2.

THEOREM VIII.

The ſquare made upon the ſide (AC) *ſubtending the right-angle of a plane triangle* (ABC), *is equal to both the ſquares* (BE, BG) *made upon the ſides* (AB, BC) *containing that angle.*



Let the ſides of the ſquares BE, BG be produced to meet each other in L and D; in which take KL and IG each equal to AE (or AB); and let CI, IK, and KA be drawn.

Since ABH and FBC (which are continued right-lines [i]) are equal to each other [k], EL, DG, ED, and LG will be all equal among themſelves [l]; and ſo the angles E, D, G and L being all right ones [m], EDGL will be a ſquare, and conſequently ACIK a ſquare likewiſe [n]. Now, if from the ſquare DL, the four equal [o] triangles ADC, CGI, ILK, and KEA be taken away, there will remain the ſquare AI: and, if from the ſame DL, the two equal [p] parallelograms DB, BL (which are equal to the ſaid four triangles, becauſe DB = two of them [l]) be taken away; then there will remain the two ſquares BE and BG. Conſequently the ſquare AI is = the two ſquares BE and BG [q].

[i] 2. 1. [k] Ax. 4. [l] 24. 1. [m] Hyp. & 5. 1. [n] 28. 1. [o] Ax. 10. [p] 1. 2. [q] Ax. 5.

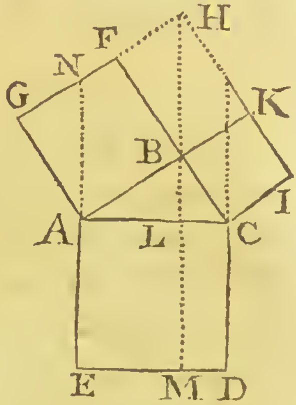
The ſame demonſtrated otherwiſe.

Let AD be the ſquare on the hypothenuſe AC, and BG, BI the two ſquares on the ſides AB and

D BC:

BC: let MBH be parallel to AE, meeting GF (produced) in H; and let EA be produced to meet GH in N.

[r] Ax. 7. If from the equal [r] angles GAB, CAN, the angle NAB, common to both, be taken away, there will remain NAG
[s] Ax. 5. = BAC [s]; whence, as the angle G is alſo = ABC [r], and the
[t] Def. 26. ſide AG = AB [t], the ſides AN and AC (= AE) are likewiſe
[u] 15. 1. equal [u]; and therefore the parallelogram AM = the paralle-
[w] Cor. to 2. 2. logram AH [w]; which laſt, and
[x] 2. 2. conſequently the former, is equal to the ſquare BG [x] ſtanding on the ſame baſe AB, and between the ſame parallels. By the ſame argument, the parallelogram CM is = the ſquare BI: and, conſequently, the ſquare AD (= AM + CM) = both the
[y] Ax. 4. ſquares BG and BI [y].



COROLLARY.

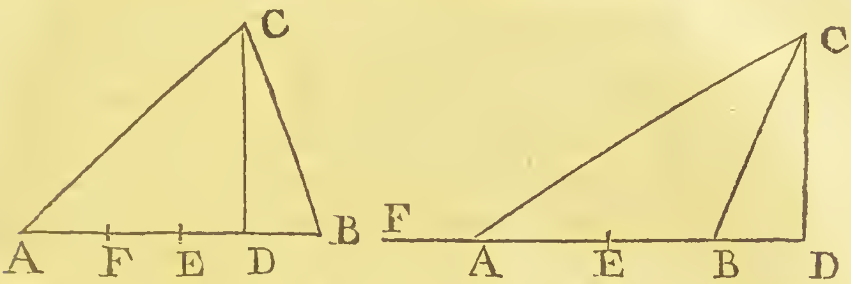
Hence, the ſquare upon either of the ſides including the right angle, is equal to the difference of the ſquares of the hypothenuſe and the other
[z] Ax. 5. ſide [z]; or, equal to a rectangle contained under the ſum and difference of the hypothenuſe and the other
[a] 7. 2. ſide [a].

THEOREM IX.

The difference of the ſquares of the two ſides (AC, BC) *of any triangle* (ABC) *is equal to the difference of the ſquares of the two lines, or diſtances* (AD, BD) *included between the extremes of the baſe* (AB) *and the perpendicnlar* (CD) *of the triangle.*

For, ſince $AC^2 = DC^2 + AD^2$, and $BC^2 = DC^2 + BD^2$ *(by the precedent)*, it is evident that the difference

difference of AC^2 and BC^2 will be equal to the difference between $DC^2 + AD^2$ and [b] $DC^2 + BD^2$, or [b] Ax. 5.



between AD^2 and [c] BD^2, by taking away DC^2, [c] Ax. 6. common, from both.

COROLLARY I.

Since the rectangle under the ſum and difference of any two unequal lines, is equal to the difference of their ſquares [d], it follows, that the difference of [d] 7. 2. the ſquares (or the rectangle under the ſum and difference) of the two ſides of any triangle, is equal to the rectangle under the ſum and difference of the diſtances included between the perpendicular and the two extremes of the baſe.

COROLLARY II.

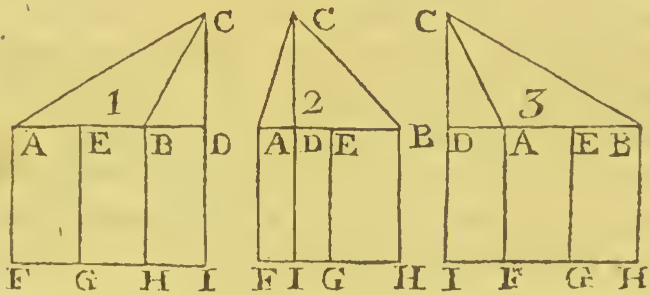
It follows, moreover, that the difference of the ſquares (or the rectangle under the ſum and difference) of the two ſides of a triangle, is equal to twice a rectangle under the whole baſe, and the diſtance of the perpendicular from the middle of the baſe.

For, let E be the middle of the baſe, and let EF be made $=$ ED; then AF being $=$ BD [e], the ex- [e] Ax. 5. ceſs of AD above BD (or AF) will (*in Fig.* 1.) be $=$ DF $=$ 2DE; therefore the rectangle under the ſum and difference of AD and BD ($=$ [g] $AC^2 - BC^2$) is $= AB \times 2DE$, [g] 9. 2. Again (*in Fig.* 2.) $AD + BD$ being $= AD + AF$ [h] $= FD = 2ED$, and [h] Ax. 4. $AD - BD = AB$, we have, alſo, in this caſe [g], $AC^2 - BC^2 = AB \times 2DE$.

THEOREM X.

The square of one side (AC) *of a triangle* (ABC) *is greater, or less than the sum of the squares of the base* (AB) *and of the other side* (BC), *by a double rectangle under the whole base* (AB) *and the distance* (AD) *of the perpendicular from the angle* (B) *opposite to the side first mentioned; that is, greater, when the perpendicular falls beyond the said angle (as in Fig 1.); but less, when it falls on the contrary side (as in Fig 2. and 3.)*

[l] 1. 2. Let the square ABHF, on the base AB, be divided into two equal [l] rectangles EF and EH by



the line EG, bisecting AB in E; and let the perpendicular CD be continued out to meet FH (produced) in I.

[l] Cor. 2. to 9. 2.
[m] Ax. 3.
[n] Ax. 5.

In Fig. 1. $AC^2 - BC^2$ = twice the rectangle EI [l] = 2EH + 2BI [m] = AH (AB^2) + 2BI (2AB × BD); therefore, if from the first and last of these equal quantities, AB^2 be taken away, then AC^2 *less* both BC^2 and $AB^2 = 2AB \times BD$ [n].

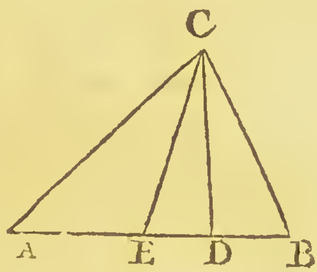
[o] Ax. 4.

In Fig. 2. *and* 3. $BC^2 - AC^2 = 2EI$ [l] $= 2BI - 2BG$ [m] $= 2AB \times BD - AB^2$; and so, by adding AB^2 to the first and last of these equal quantities, we have here $AB^2 + BC^2 - AC^2 = 2AB \times BD$ [o].

THEO-

THEOREM XI.

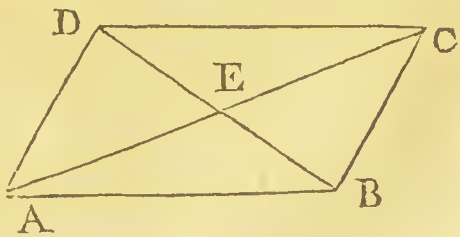
The double of the ſquare of a line (CE) *drawn from the vertex to the middle of the baſe of any triangle* (ABC), *together with double of the ſquare of the ſemi-baſe* (AE), *is equal to the ſquares of both the ſides* (AC, BC) *taken together.*



For, let CD be perpendicular to AB: then, becauſe (*by the precedent*) AC^2 exceeds the ſum of the two ſquares AE^2 and CE^2 (or BE^2 and CE^2) by the double rectangle $2AE \times ED$ (or $2BE \times ED$); and becauſe BC^2 is leſs than the ſame ſum by the ſame double rectangle; it is manifeſt that both AC^2 and BC^2 together muſt be equal to that ſum twice taken; the exceſs on the one part making up the defect on the other.

THEOREM XII.

The two diagonals (AEC, BED) *of a parallelogram* (ABCD) *biſect each other; and the ſum of their ſquares is equal to the ſum of the ſquares of all the four ſides of the parallelogram.*

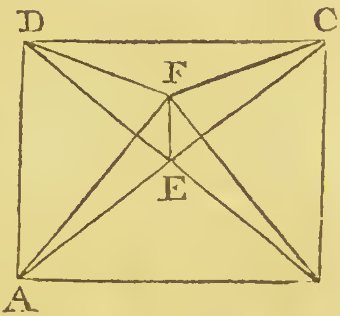


For, the triangles AEB, DEC being equiangular [p], and having AB = DC [q], will alſo have AE = CE, and BE = DE [r]. Moreover, becauſe $2AE^2 + 2ED^2 =$ [s] $AD^2 + CD^2$, by taking the double of theſe, we have $4AE^2$ ([t] AC^2) $+ 4ED^2$ (DB^2) $=$ [u] $AD^2 + BC^2 + CD^2 + AB^2$.

p 3. and 7. 1.
q 24. 1.
r 15. 1.
s 11. 2.
t Cor. 1. to 6. 2.
u Ax. 4. and 24. 1.

THEOREM XIII.

If from any point (F), *to the four angles of a rectangle* (ABCD) *four lines be drawn; the sums of the squares of those drawn to the opposite angles will be equal (I say, that* $FA^2 + FC^2 = FB^2 + FD^2$).



For, let the diagonals AC and BD be drawn, bisecting
x 12. 2. each other in E[x], and let E, F be joined; then the triangles ABC, BAD being
w 24. 1. and Ax. 10. equal in all respects[w], thence will $AE(\frac{1}{2}AC) = DE(\frac{1}{2}DB)$.
y 11. 2. But $FA^2 + FC^2 =$[y] $2AE^2 (2DE^2) + 2EF^2 =$[y] $FB^2 + FD^2$.

End of the Second Book.

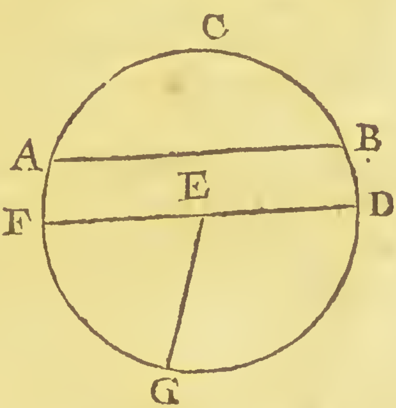
ELEMENTS

OF

GEOMETRY.

BOOK III.

DEFINITIONS.



1. ANY right-line FD, paffing through E the center of a circle, and terminating in the circumference at both ends, is called a Diameter.

2. An arch of a circle, is any portion of the periphery, or circumference, as ACB.

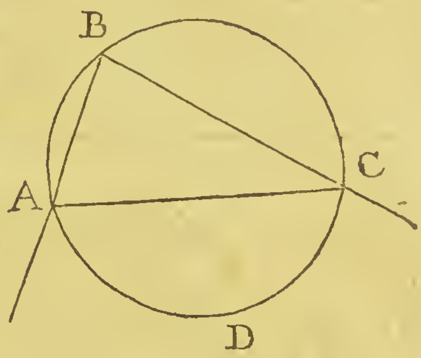
3. The chord, or fubtenfe of an arch ACB, is a right-line AB joining the two extremes of that arch.

4. A femi-circle is a figure contained under any diameter and either part of the circumference cut off by that diameter.

5. A ſegment of a circle is a figure contained under an arch ACB and its chord AB.

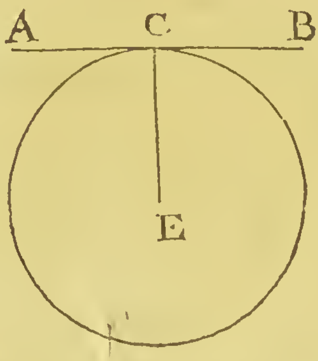
6. A Sector of a circle is a figure contained under two right-lines EF, EG, drawn from the center to the circumference, and the arch FG included betwixt them. When the two lines EF, EG, ſtand perpendicular to each other, then the Sector is called a quadrant.

7. An angle ABC is ſaid to be in a ſegment of a circle ABC, when, being in the periphery thereof the right-lines BA, BC by which it is formed, paſs through the extremes of the chord AC bounding that ſegment.



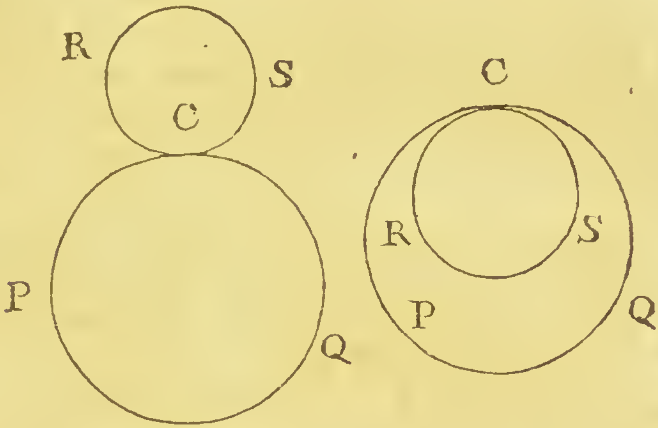
8. An angle ABC in the periphery, comprehended by two right-lines BA, BC, including an arch of the circle, ADC, is ſaid to ſtand upon that arch.

9. A right-line AB is ſaid to touch a circle, when, paſſing through a point (C) in the circumference thereof, it cutteth off no part of the circle.



10. Two

10. Two circles (PCQ, RCS) are faid to touch each other, when the circumferences of both pafs



through one point (C) and yet do not cut each other.

11. Two circles, in the fame plane, are faid to cut one another, when they fall partly within, and partly without each other; or, when their circumferences cut each other.

12. A right-line is faid to be applied, or infcribed in a circle, when both its extremes are in the periphery thereof.

13. A right-lined figure is faid to be infcribed in a circle, when all its angles are in the circumference of the circle.

14. A circle is faid to be defcribed about a right-lined figure, when the periphery of the circle paffes through all the angles of that figure.

15. A right-lined figure is faid to be defcribed about a circle, when all the fides thereof touch the circle.

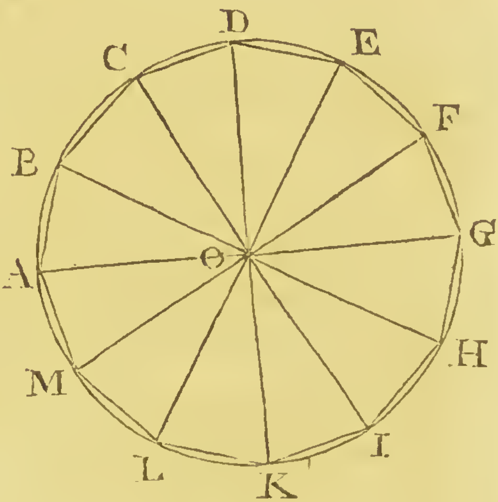
16. A

16. A circle is ſaid to be inſcribed in a right-lined figure, when it is touched by all the ſides of the right-lined figure.

17. A right-lined figure is ſaid to be inſcribed in a right-lined figure, when all the angles of the former are ſituate in the ſides of the latter.

THEOREM I.

If the ſides (AB, BC, CD, &c.) *of a polygon inſcribed in a circle, be equal, the angles* (AOB, BOC, COD, &c.) *at the center of the circle, ſubtended by them, will likewiſe be equal.*



For AO, BO, CO &c. being equal to each other [a], as well as AB, BC, CD &c. the triangles AOB, BOC, COD, are mutually equilateral; and therefore have all the angles AOB, BOC &c. equal to each other [b].

[a] Def. 33. of 1.

[b] 14. 1.

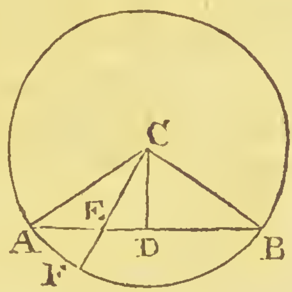
SCHOLIUM.

On this propoſition depends the diviſion of mathematical inſtruments for taking and meaſuring of angles. For, if, by repeated trials, or any other means, the circumference of a circle deſcribed about a center O, be divided into any number of parts AB, BC, CD &c. ſo that the chords be equal; then it is evident, from hence, that all the angles AOB, BOC, COD &c. which make up the four right-angles AOD, DOG, GOK, KOA at the center, will alſo be equal to each other, let the

the radius OA of the *instrument* be what it will.— In the division of the circle for practical uses, the number of parts into which the circumference is thus divided, or the number of equal angles at the center, is 360; which equal angles are called *degrees*; so that a right angle, consisting of 90 of these equal angles, is said to be an angle of 90 degrees; every angle being denominated, from the degrees and parts of a degree, contained therein; each degree being conceived to be subdivided into 60 equal parts, called *minutes*; each minute again into 60 equal parts, called *seconds*; and so on to thirds, fourths, fifths, *&c.* at pleasure.

THEOREM II.

Any chord (AB) *of a circle, falls wholly within the same: and a perpendicular* (CD) *let fall thereon, from the center of the circle, will divide it into two equal parts.*



Let C, A, and C, B be joined; and thro' any point E in the chord AB, let the right-line CEF be drawn, meeting the circumference in F.

It is evident, because CA = CB[c], that these equal lines are on different sides of the perpendicular CD[d]; and so, CE being ⊐ CA or CF[d], the point E (take it where you will in the line AB) and consequently the line AB itself, will fall within the circle[e]. Moreover, because the triangles ACD, BCD have CA = CB and CD common, thence will AD be also = BD[f].

[c] Def. 33. 1.
[d] 20. 1.
[e] Ax. 2.
[f] 16. 1.

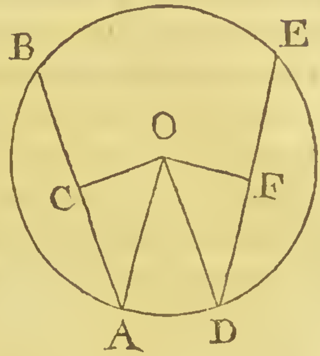
COROLLARY.

Hence a line bisecting any chord at right-angles, passes thro' the center of the circle.

THEO-

THEOREM III.

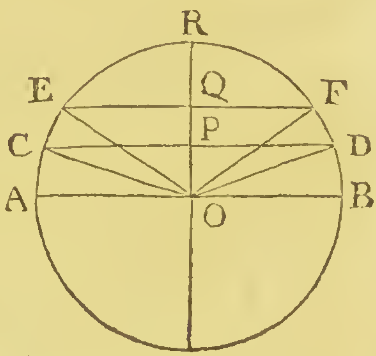
Any two chords (AB, DE) *equally diſtant from the center* (O) *of a circle, are equal to each other.*



Let the perpendiculars OF, OC be drawn, and let O, D
[e] Hyp. and O, A be joined. Becauſe
[f] Def. 33. of 1. OF = OC [e], OD = OA [f],
[g] Conſtr. and F and C are both right-
[h] 16. 1. angles [g], therefore is DF =
[i] 2. 3. AC [h], and conſequently DE
[k] Ax. 4. 1. = 2DF [i] = 2AC [k] = AB [i].

THEOREM IV.

In a circle (AEFB) *the greateſt line* (AB) *is the diameter*; *and, of all others terminating in the circumference, that* (CD) *which is neareſt the center* (O), *is greater than any other* (EF) *further from it.*



1. Draw OC and OD; then it will appear that AB
[m] 19. 1. (or OC + OD) $\sqsubset$ CD [m].

2. Let OP be the diſtance of CD from the center, and OQ that of EF, both taken in the ſame radius OR; Draw OE and OF. Becauſe the triangles
[n] Def. 33. 1. DOC, OFE, have two ſides equal each to each [n], and have the contained angle DOC $\sqsubset$ the contained
[o] Ax. 2. angle FOE [o]; therefore, alſo, will the baſe DC be
[p] 21. 1. greater than the baſe FE [p]; and, conſequently, greater than any other chord at the ſame diſtance,
[q] 3. 3. with EF [q].

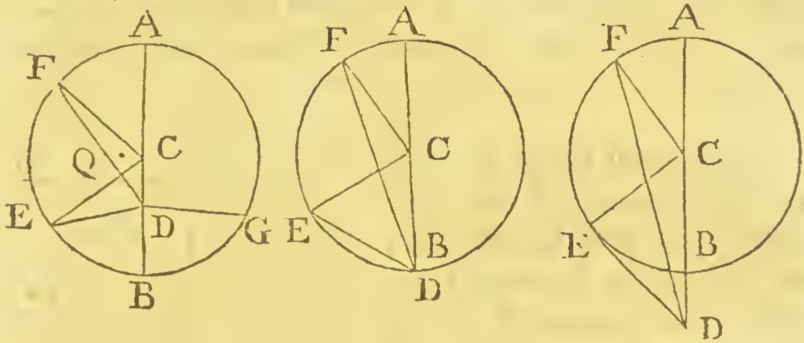
COROLLARY.

Hence, a right-line greater than the diameter, drawn from any point within a circle, will cut the circumference.

THEO-

THEOREM V.

If to the circumference of a circle (AFEB), *from any point* (D) *which is not the center, right-lines* (DA, DF, DE) *be drawn, the greatest of all* (DA) *shall be that which passes through the center* (C); *and, of the rest, that* (DF) *whose other extreme* (F) *is placed nearest, in the circumference, to the extreme* (A) *of the greatest, will exceed any other* (DE) *whose extreme* (E) *is at a greater distance,*



From the center C, let CE and CF be drawn.

1. AD ($=$ DC $+$ CF [r]) $\sqsubset$ DF [s]. [r] Ax. 4

2. Since DC is common CF $=$ CE, and DCF [s] 19. 1.
$\sqsubset$ DCE [t], therefore is DF $\sqsubset$ DE [u]. [t] Ax. 2. [u] 21. 1.

COROLLARY I.

Because no two lines, DE, DF, drawn from D, on the same side of the diameter AB, can be equal to each other [w], three equal right-lines cannot possibly be drawn from the periphery to any point, besides the center of the circle: and, therefore, if from a point in any circle, three equal right-lines can be drawn to the periphery, that point is the center of the circle. [w] 5. 3.

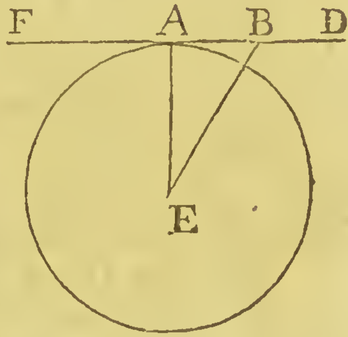
COROLLARY II.

Hence it also follows, that no circle can be described to cut another FBG in more points than two: for, if it were possible to cut it in three points G, E, F, then right-lines drawn from the center Q,

[x] Def. 33. 1. Q, to thofe points, would be all equal [x], which is fhewn to be impoffible [y], unlefs when the center
[y] Cor. 1. to 4. 3. Q coincides with C; and then the circles themfelves will neither cut, nor touch, but coincide, and become one circle [x].

THEOREM VI.

A right-line (FD) *drawn through any point* (A) *in the circumference of a circle, at right-angles to the radius* (EA) *terminating in that point, will touch the circle.*

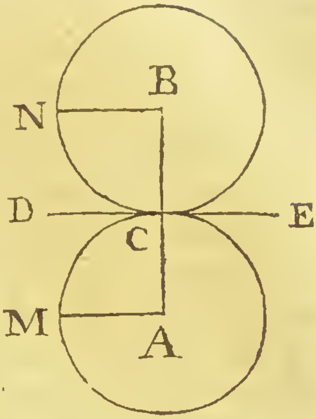


From any point in FD, to the center E, let the right-line BE be drawn; which being greater than
[a] 20. 1. AE [a], the point B muft, neceffarily, fall out of the
[b] Def. 33. and Ax. 2. of 1. circle [b]: and therefore, as the fame argument holds good with regard to every other point in the line FD (except A), it is manifeft that this line cuts off no part of the circle, but touches it, in one point only.

THEOREM VII.

If the diftance (AB) *of the centers of two circles, be equal to the fum of the two femi-diameters* (AM, BN), *the circles will touch each other, outwardly; and the right-line* (AB) *joining their centers, will pafs through the point of contact.*

In AB, take AC = AM, and let DCE be drawn perpendicular to AB: then, BC being alſo = BN [c], the circumferences of both circles will paſs through the point C [d]: but the right-line DE (*by the precedent*) falls wholly above the one, and wholly below the other; therefore the circles themſelves fall wholly without each other, and touch in one point C only.



[c] Conſtr. and Ax. 5.

[d] Def. 33. 1.

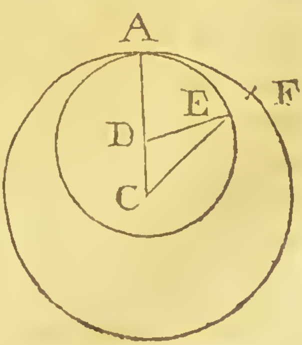
COROLLARY.

Hence, if the centers of two circles be placed at a diſtance, from one another, leſs than the ſum of the two ſemi-diameters, a part, at leaſt, of the one will be contained within the other: but, if the diſtance be greater than that ſum, the two circles will then neither touch, nor cut each other.

THEOREM VIII.

If the diſtance (CD) *of the centers of two circles* (CAF, DAE) *be equal to the difference of the two ſemi-diameters* (CA, DE), *then will thoſe circles touch inwardly; and that radius* (CA) *of the greater, which is drawn through the center* (D) *of the leſſer, will meet the two peripheries in the point of contact.*

From any point E in the circumference of the leſſer, to the two centers, let EC and ED be drawn. Becauſe CA exceeds DE by the line DC [e], or becauſe DE + DC = CA [f] = DA + DC [g], therefore is DA = DE [h]; and ſo the circumference of the circle D



[e] Hyp.

[f] Ax. 4.

[g] Ax. 3.

[h] Ax. 5.

likewiſe

likewise passes through A: but CA is greater than CE[i]: therefore every point in the periphery of the circle D (except A only) falls within the circle C: *which was to be demonstrated.*

[i] 5. 3.

COROLLARY I.

Hence, if the centers of two circles be placed at a distance from each other, greater than the difference of the two semi-diameters, a part, at least, of the one will fall without the other; but, if the distance be less than that difference, the lesser circle will then be contained wholly in the greater, but without touching it.

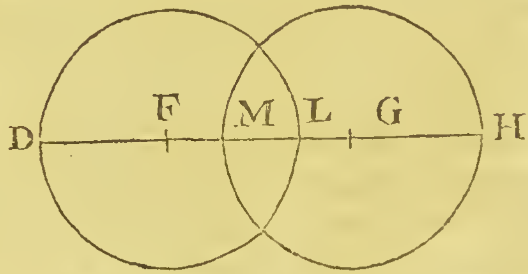
COROLLARY II.

Hence, and from *the precedent*, it likewise appears, that if two circles touch, either inwardly or outwardly, a right-line, drawn through their two centers, will also pass through the point of contact: because they can only touch, when the distance of their centers is equal to the sum, or to the difference of their semi-diameters[k].

[k] Cor. of 7. and Cor. 1. of 8.

THEOREM IX.

If the distance of the centers (F, G) *of two circles* (DL, MH) *be less than the sum, and greater than the difference of the two semi diameters* (FL, GM), *those circles will cut each other.*



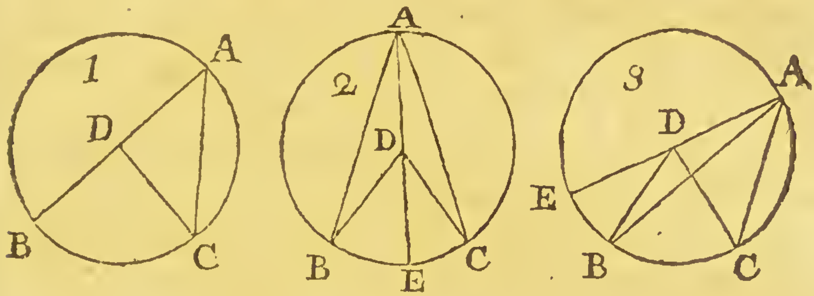
For, since the distance of the two centers is supposed less than the sum of the semi-diameters, a part of the one circle MH, falls within the other DL[l]; and since that distance is greater than the difference of those semi-diameters, a part of the same circle MH also falls without the circle DL[m]: *which was to be proved*[n].

[l] Cor. to 7. 3.
[m] Cor. to 8. 3.
[n] Def. 11. of 3.

THEO-

THEOREM X.

The angle (BDC) *at the center of a circle, is double to the angle* (BAC) *at the circumference, when both angles ſtand upon the ſame arch* (BC).



Let the diameter ADE be drawn.

In the firſt caſe (where AB paſſes through the center) BDC = A + C [o] = 2A [p]. [o] 9. 1. [p] 12. 1.

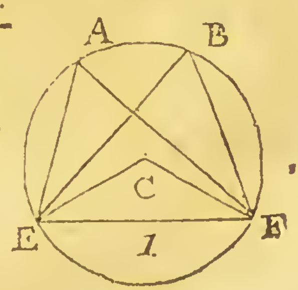
In the ſecond caſe, BDE = 2BAE, (*by caſe* 1.); to which adding CDE = 2CAE, we have BDC = 2BAC [q]. [q] Ax. 4.

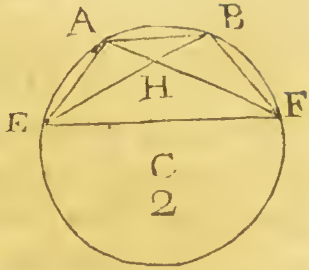
In the third caſe, CDE = 2CAE (*by caſe* 1.) from whence ſubtracting BDE = 2BAE, there remains BDC = 2BAC [r]. [r] Ax. 5.

THEOREM XI.

All angles (EAF, EBF) *in the ſame ſegment* (EABF) *of a circle, are equal to each other.*

Case I. *If the ſegment be greater than a ſemi-circle*; from the center C draw CE and CF; then EAF and EBF being each of them = to half ECF [s], they muſt neceſſarily be equal to each other. [s] 10. 3.

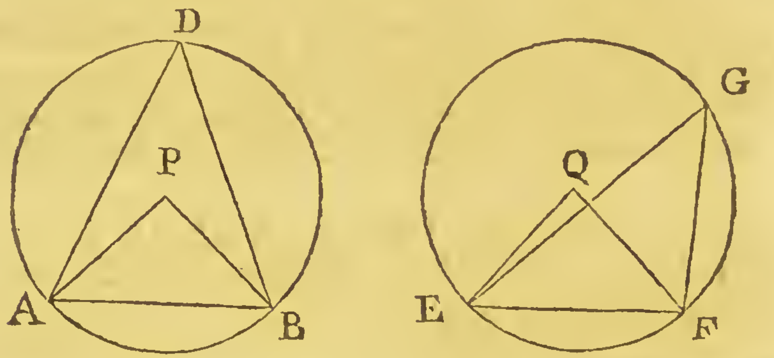




CASE II. *If the segment be less than a semi-circle*; let H be the intersection of EB and AF: then the triangles AEH and BFH, having the angle
[t] 3. 1. AHE = BHF [t], and AEH = BFH (*by case* 1.) they will
[u] Cor. 1. to 10. 1. also have EAH = FBH [u].

THEOREM XII.

Angles (D, G) *in the circumferences, standing upon equal subtenses* (AB, EF) *of circles having equal diameters, are equal to each other. And the subtenses of equal angles, in the circumference of circles having equal diameters, are also equal.*



From the centers P, and Q, let PA, PB, QE, QF be drawn.

[w] Hyp. 1. Hyp. Since AB = EF [w], and AP = BP [x] =
[x] Def. 33. of 1. EQ = FQ [w]; therefore is P = Q [y], and consequently D ($= \frac{1}{2}$ P [z] $= \frac{1}{2}$ Q) = G.
[y] 14. 1.
[z] 10. 3. 2. Hyp. Because D = G, therefore P = Q [z]; whence, PA being = QE, and PB = QF [w], AB
[a] Ax. 10. of 1. will also be = EF [a].

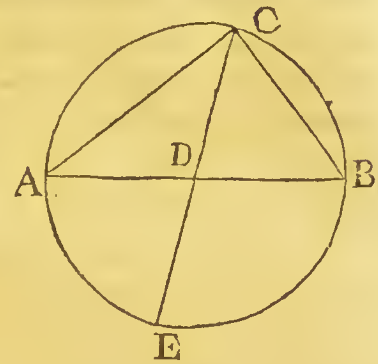
COROLLARY.

Hence angles in the circumference, standing upon equal chords of the same circle, are equal.

THEO-

THEOREM XIII.

The angle (ACB) *in a semi-circle, is a right angle.*



Let the diameter CDE be drawn.

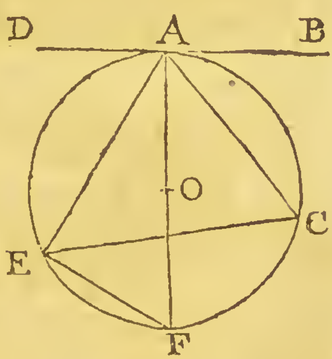
Becaufe ACD = half ADE, and BCD = half BDE [b], therefore is ACD + BCD (= ACB) = half of ADE and BDE [c] = half two right angles [d] = one right angle.

[b] 10. 3.
[c] Ax. 4. 1.
[d] 1. 1.

THEOREM XIV.

The angle (CAB) *included by a tangent to a circle and a chord* (AC) *drawn from the point of contact* (A), *is equal to the angle* (AEC) *in the alternate segment.*

Let the diameter AOF be drawn, and E, F be joined.

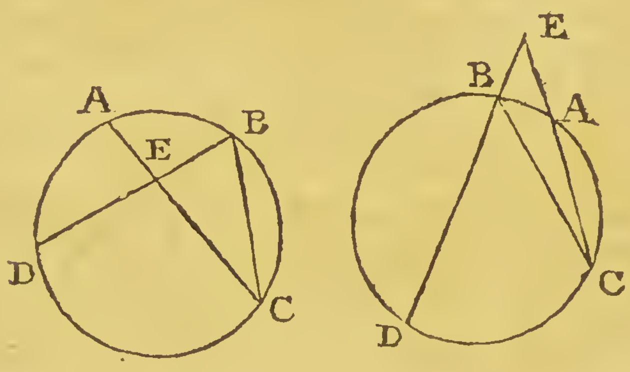


The line DB falling wholly above the circle [e], OA is the leaft line that can be drawn to it from the center O [f]; and OAB is therefore a right-angle [g]: but FEA is alfo a right angle [h]: therefore, if from thefe equal angles, the equal [i] angles FAC, FEC (ftanding on the fame arch FC) be taken away, there will remain BAC = AEC [k].

[e] Def. 9. 3.
[f] Def. 31. 1, and Ax. 2.
[g] 20. 1.
[h] 13. 3.
[i] 11. 3.
[k] Ax. 5.

THEOREM XV.

The angle (DEC) *made by two lines* (DEB, CEA) *interſecting each other within, or without a circle, is, in the former caſe, equal to the ſum, and in the latter, equal to the difference, of two angles in the circumference, ſtanding on the two arcs* (DC, AB) *intercepted by thoſe lines.*



Let the chord CB be drawn.

[g] 9. 1. Then DEC = DBC + ACB [g], *in the firſt caſe.*

[h] 9. 1. and Ax. 5. And DEC = DBC — ACB [h], *in the ſecond caſe.*

COROLLARY.

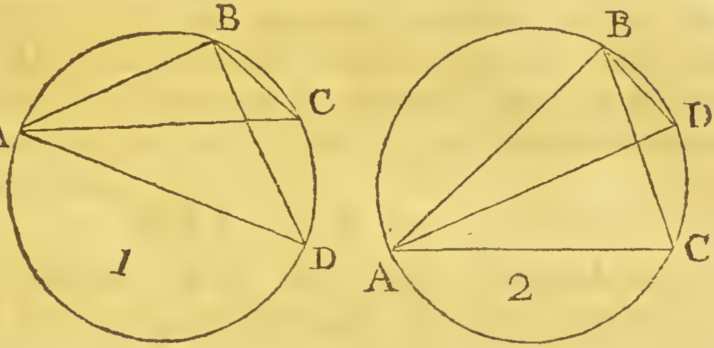
Hence an angle (E) formed below, or above the circumference of a circle, is greater, or leſs than an angle in the circumference, ſtanding on the ſame arch.

THEOREM XVI.

The vertical angle (ABC) *of any oblique-angled triangle* (ACB) *inſcribed in a circle* (ABCD) *is greater, or leſs than a right-angle, by the angle* (CAD) *com-*

prehended

prehended under the base (AC) *and the diameter* (AD) *drawn from the extremity of the base.*

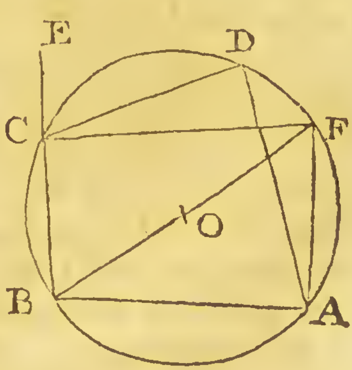


For, BD being drawn, ABD will be a right-angle [i], and CAD = CBD [k]; therefore, in the first case, ABC = right-angle + CAD [l]; and in the second, ABC = right-angle — CAD [m].

[i] 13. 3.
[k] 11. 3.
[l] Ax. 4.
[m] Ax. 5.

THEOREM XVII.

If any side (BC) *of a quadrilateral* (ABCD) *inscribed in a circle, be produced out of the circle, the external angle* (ECD) *will be equal to the opposite, internal angle* (BAD).



Let the diameter BF be drawn, and let AF and CF be joined: then the angle BAF being a right angle (= BCF) = ECF [n], and DAF also = DCF [o], standing both on the same arch DF; thence will the remainders BAD and ECD be also equal [q].

[n] 13. 3.
[o] 11. 3.
[q] Ax. 5.

 COROL-

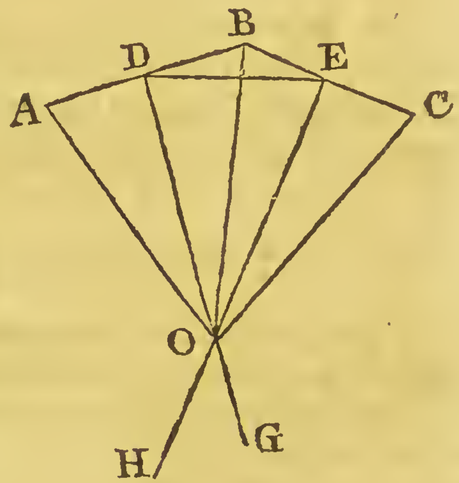
COROLLARY.

Hence the oppofite angles BAD, BCD of any quadrilateral infcribed in a circle, are together, equal to two right-angles. For, fince BAD = ECD, therefore is BAD+BCD = ECD + BCD [r] = two right angles [s].

[r] Ax. 4.
[s] 1. 1.

THEOREM XVIII.

Through any three points (A, B, C) *not fituate in the fame right line, the circumference of a circle may be defcribed.*



Draw AB and BC, which let be bifected by the perpendiculars DG and EH, infinitely produced on that fide of AB or BC, on which the angle ABC is formed.

Thefe perpendiculars, I fay, will interfect each other; and the point of interfection O, will be the center of the circle.

For, if DE be drawn, it is plain, that the angles GDE, HED are lefs than two right-angles [t]; therefore DG, EH, not being parallels [u], they will meet each other [w]. Hence, if from the point of interfection O, the right lines OA, OB, OC be drawn, the triangles ADO, BDO, having two fides equal, each to each [x], and the angles ADO, BDO, contained by them, equal [y], will likewife have AO=BO [z]. After the very fame manner is CO = BO; therefore AO = BO = CO [a]: whence the circumference of a circle defcribed from the center O, at the

[t] Ax. 2.
[u] Cor. 2. to 7. 1.
[w] Def. 13. of 1.
[x] Conftr.
[y] Ax. 7.
[z] Ax. 10.
[a] Ax. 1.

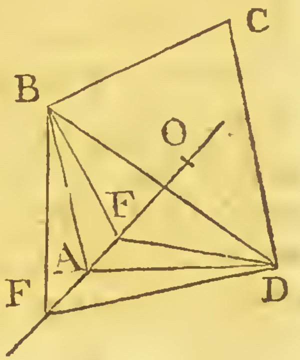
the diſtance of AO, will alſo paſs through B and C[b]. [b] Def. 33. 1.

SCHOLIUM.

Hence the method of deſcribing the circumference of a circle through three given points, is manifeſt.

THEOREM XIX.

*If the oppoſite angles (*BAD, BCD*) of a quadrilateral (*ABCD*) be equal to two right-angles, a circle may be deſcribed about that quadrilateral.*

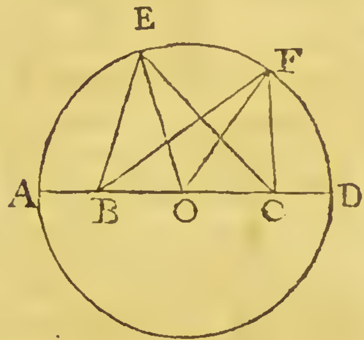


For the circumference of a circle may be deſcribed thro'any three points B, C, D, (*by the precedent.*) But, if you deny that it paſſes thro' A; then, thro' the center O, let OAF be drawn, and let it (if poſſible) paſs through ſome other point F in the line OAF, (for it muſt cut this line ſomewhere [c]); alſo let BF and DF be drawn. Becauſe BFD + BCD = two right-angles [d] = BAD + BCD [e]; therefore muſt BFD = BAD [f]: *which is impoſſible* [g]. Therefore the circumference of the circle deſcribed through B, C and D, muſt alſo paſs through A.

[c] Cor. to 4. 3.
[d] Cor. to 17. 3.
[e] Hyp.
[f] Ax. 5.
[g] 23. 1.

THEOREM XX.

If from two points (B, C) *in the same diameter* (AD), *equally distant from the center* (O) *of a circle, right-lines* (BE, CE; BF, CF) *be drawn to meet, two by two, in the circumference; the sum of the squares of any two corresponding ones, will be equal to the sum of the squares of any other two, meeting in like manner.*

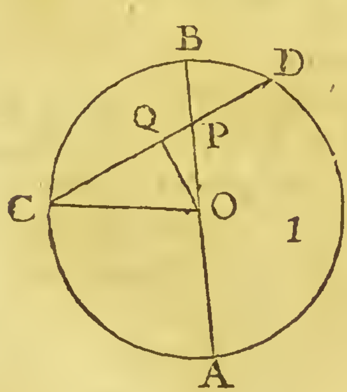


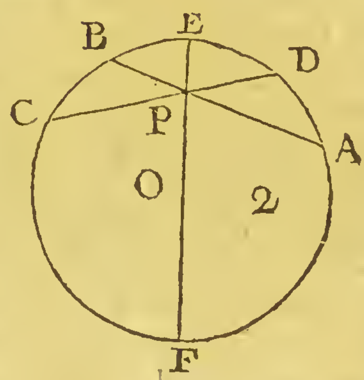
For, if OE and OF be drawn; then will $BE^2 + CE^2 = {}^h\ 2BO^2 + 2OE^2$ $(2OF^2 = {}^h\ BF^2 + CF^2$.

h 11. 2.

THEOREM XXI.

If two lines (AB, CD), *terminated by the periphery on both sides, cut each other within a circle, the rectangle* (AP × BP) *contained under the parts of the one, will be equal to the rectangle* (CP × DP) *contained under the parts of the other.*





CASE I. If one of the two lines (AB) passes through the center O; then let OQ be drawn perpendicular to the other CD, and let OC be joined.

It

It is plain, because $QD = QC$ [i], that DP is equal to the difference of the segments CQ and PQ [k]: but the rectangle under the sum and difference of the two sides OC, OP, of any triangle COP, is equal to the rectangle under the whole base CP, and the difference of its two segments [l]; therefore, the sum of the two sides OC, OP being $(= OA + OP) = AP$, and their difference $(= OB - OP) = BP$, thence is the rectangle contained under AP and BP equal to the rectangle contained under CP and DP.

[i] 2. 3.

[k] Ax. 5.

[l] Cor. 1. to 9. 2.

Case II. If neither of the two lines pass through the center; let the diameter EPF be drawn; then, *by the preceding case*, $AP \times BP = FP \times EP = CP \times DP$.

THEOREM XXII.

If from two points (A, C) *in the circumference of a circle, two lines* (AP, CP) *be drawn, to pass through and meet without the circle; the rectangle* $(AP \times BP)$ *contained under the whole and the external part of the one, will be equal to the rectangle* $(CP \times DP)$ *contained under the whole and the external part of the other.*

Through the center O, let PF be drawn, meeting the circumference in E and F; let OQ be perpendicular to AP, and let A, O be joined.

Then, (*by Cor.* 1. *to* 9. 2.) the rectangle contained under PF $(= PO + OA)$ and PE $(= PO - OA)$ is = the rectangle contained under AP and PB. After the very same manner, $PF \times PE = CP \times DP$: therefore $AP \times BP = CP \times DP$ [m].

[m] Ax. 1.

COROL.

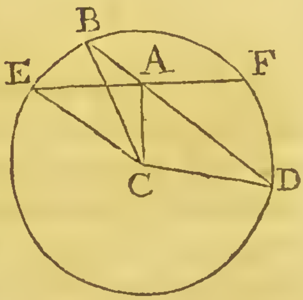
COROLLARY I.

Hence, if PS be a tangent at S, and the radius OS be drawn; then, PF being = the sum of PO and OS, and PE = their difference; it follows, that $PS^2 = PF \times PE$ [o] $= PC \times PD$.

[o] Cor. to 8. 2.

THEOREM XXIII.

If from the center (C) *of a circle, to a point* (A) *in any chord* (BD), *a line* (CA) *be drawn; the square of that line, together with the rectangle contained under the two parts of the chord, will be equal to a square made upon the radius of the circle.*



Let EAF be another chord, perpendicular to CA, and let C, E be joined.

Since AF = AE [q], thence will AE^2 [r] $= AE \times AF$ [s] $= AB \times AD$ [t]; to which equal quantities adding AC^2, we have $CE^2 = AB \times AD + AC^2$.

[q] 2. 3.
[r] 1. 2.
[s] 21. 3.
[t] 8. 2.

COROLLARY.

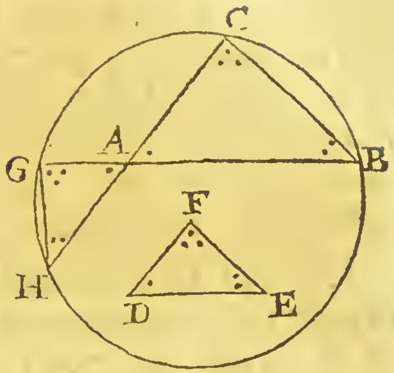
Hence the square of a line (AC) drawn from any point in the base of an isosceles triangle (BCD) to the opposite angle, together with the rectangle of the parts of the base, is equal to a square made upon one of the equal sides of the triangle.

THEOREM XXIV.

The rectangles contained under the corresponding sides of the equiangular triangles (ABC, DEF) *taken alternately, are equal.*

I say, if A = D, B = E and C = F, then will $AB \times DF = AC \times DE$.

In



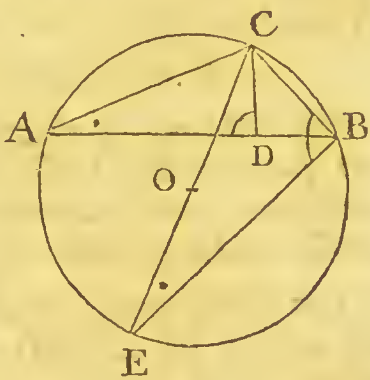
In BA produced, let AG be taken = DF; let GCB be the circumference of a circle paſſing through the three points B, C, G[w], meeting CA produced in H; and let G, H be joined. [w] 18. 3.

Becauſe the angle H = B[y] = E[z], HAG = [a] BAC = D[z], and AG = DF[z], thence is AH = DE[b]; and therefore AC × DE = AC × AH[c] = AB × AG[d] = AB × DF[c].

[y] 11. 3. [z] Hyp. [a] 3. 1. [b] 15. 1. [c] 1. 2. [d] 21. 3.

THEOREM XXV.

The rectangle under the two ſides (AC, BC) *of any triangle* (ABC), *is equal to the rectangle under the perpendicular* (CD) *to the baſe thereof, and the diameter* (CE) *of the circumſcribing circle.*



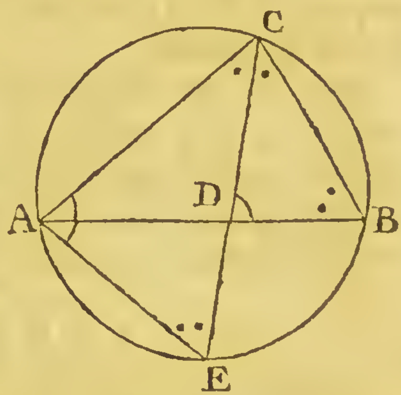
For, B, E being joined, the angles A, E will be equal[f], and ADC, EBC both right-angles[g]; and, conſequently the triangles ACD, ECB equiangular[h]: therefore AC, EC; CD, CB being correſponding ſides, oppoſed to equal angles, the rectangle AC × CB, contained under the firſt and laſt of them, will be equal to the rectangle EC × CD contained under the other two[i].

[f] 11. 3. [g] Hyp. and 13. 3. [h] Cor. 1. to 10. 1. [i] 24. 3.

THEO-

THEOREM XXVI.

The ſquare of a line (CD) *biſecting any angle* (C) *of a triangle* (ABC) *and terminating in the oppoſite ſide* (AB), *together with the rectangle* ($AD \times BD$) *under the two ſegments of that ſide, is equal to the rectangle of the two ſides including the propoſed angle.*



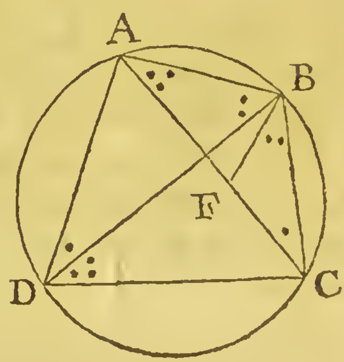
Let CD be produced to meet the circumference of a circle, deſcribed[k] through the points A, C, B, in E; and let AE be drawn.

The angles E and B, ſtanding upon the ſame ſegment AC, are equal[l]; and ACE is equal DCB (by hypotheſis); therefore the triangles AEC, DCB are equiangular[m]; whereof AC, CD; CE, CB are correſponding ſides, oppoſed to equal angles: therefore $AC \times CB = CD \times CE$[n] $= CD^2 + CD \times DE$[o] $= CD^2 + AD \times DB$[p].

[k] 18. 3.
[l] 11. 3.
[m] Cor. 1. to 10. 1.
[n] 24. 3.
[o] 5. 2.
[p] 21. 3. and Ax. 4.

THEOREM XXVII.

The rectangle of the two diagonals (AC, BD) *of any quadrilateral* (ABCD) *inſcribed in a circle, is equal to the ſum of the two rectangles* ($AB \times DC$, $AD \times BC$) *contained under the oppoſite ſides.*



Let BF be drawn, making the angle CBF = ABD, and meeting AC in F.

Becauſe the angle BCF = ADB[q], and CBF = ABD[r], the triangles CBF, DBA are equiangular[s]; and therefore, BC, BD; CF, AD, being correſponding ſides, the rectangles $BC \times AD$, and $BD \times$

[q] 11. 3.
[r] Conſtr.
[s] Cor. 1. to 10. 1.

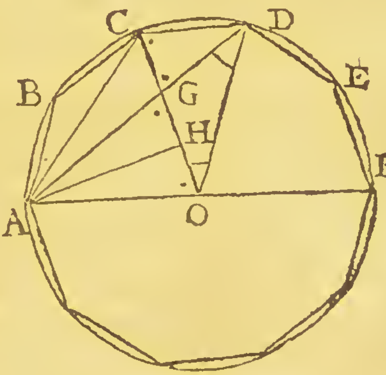
CF

CF will be equal[t]. Again, the angle ABF being $=$CBD[u], and BAF$=$BDC[w]; the triangles ABF and BDC are, likewise, equiangular; and consequently, AB, BD; AF, DC being corresponding sides, $AB \times DC = BD \times AF$[t]; to which adding $BC \times AD = BD \times CF$ (*so proved above*) we have also $AB \times DC + BC \times AD = BD \times AF + BD \times CF = BD \times AC$[x].

[t] 24. 3.
[u] Ax. 4. 1.
[w] 11. 3.
[x] 5. 2.

THEOREM XXVIII.

*If the radius of a circle (*OADF*) be so divided into two parts, that the rectangle under the whole and the one part shall be equal to the square of the other part; then will this last part be equal to the side* (CD) *of a regular decagon* (ABCDEF, &c.) *inscribed in the circle; and that line whose square is equal to the two squares, of the whole and of the same part, will be equal to the side* (AC) *of a regular pentagon inscribed in the same circle.*



Draw the radii OA, OC, OD, OF; also draw AD, cutting OC in G, and let AH be perpendicular to OG.

The triangle CDG, having the angle COD ($= \frac{1}{2}$DOF[y] $=$ OAD[z]) $=$ODA[a], is isosceles[b]: moreover the triangle AOG, having AGO ($=$ GDO $+$ DOG[c] $=$ 2DOC[n]) $=$ AOC, is likewise isosceles[d]; as is also the triangle CDG, because, CGD being $=$ AGO[e], and CDG (CDA) $=$ FAD[f], the triangles AOG, CDG are equiangular. Therefore, CD, AO; CG, GO being corresponding sides, we have $CG \times AO$ ($CG \times CO$) $= CD \times GO$[g] $= GO^2$, because GO $=$ GD

[y] 1. 3.
[z] 10. 3.
[a] 12. 1.
[b] 18. 1.
[c] 9. 1.
[d] 18. 1.
[e] 3. 1.
[f] Cor. to 12. 3.
[g] 24. 3.

h 18. 1. $=GD=DC$[h]: whence the former part of the propofition is manifeft.

i 16. 1. Again, becaufe $AG=AO$, HG will be $=HO$[i]; and fo GC being the difference of the fegments HO and HC, we have (*by Cor.* 1. *to* 9. 2.) $AC^2-AO^2=CO\times CG=OG^2$ (*as above*); and confequently $AC^2=AO^2+OG^2$.

End of the THIRD BOOK.

ELEMENTS

OF

GEOMETRY.

BOOK IV.

DEFINITIONS.

1. RATIO is the proportion which one magnitude bears to another magnitude of the same kind, with respect to quantity.

The measure, or quantity of a ratio is conceived by considering what part, or parts the magnitude referred, called the antecedent, is of the other, to which it is referred, called the consequent.

2. Three quantities, or magnitudes A, B, C, are said to be proportional, when the ratio of the first A to the second B, is the same as the ratio of the second B, to the third C. A, B, C. 2. 4. 8.

3. Four quantities A, B, C, D, are said to be proportional, when the ratio of the first A to the second B, is the same as the ratio of the third C to the fourth D. A,B,C,D. 2.4.5.10.

To

To denote that four quantities A, B, C, D *are proportional, they are uſually wrote thus,* A : B : : C : D ; *and read thus, as* A *is to* B, *ſo is* C *to* D. *But when three quantities* A, B, C *are proportional, the middle one is repeated, and they are wrote thus,* A : B : : B : C.

4. Of three proportional quantities, the middle one is ſaid to be a Mean-proportional between the other two ; and the laſt, a Third-proportional to the firſt and ſecond.

5. Of four proportional quantities, the laſt is ſaid to be a Fourth-proportional to the other three, taken in order.

A, B, C, D, E. 1. 2. 4. 8. 16.

6. Quantities are ſaid to be continually proportional (or in continual proportion) when the firſt is to the ſecond, as the ſecond to the third, as the third to the fourth, as the fourth to the fifth, and ſo on.

7. In a ſeries, or rank of quantities continually proportional, the ratio of the firſt and third, is ſaid to be duplicate to that of the firſt and ſecond ; and the ratio of the firſt and fourth, triplicate to that of the firſt and ſecond.

8. Any number of quantities, A, B, C, D being given, or propounded, the ratio of the firſt (A) to the laſt (D) is ſaid to be compounded of the ratios of the firſt to the ſecond, of the ſecond to the third, and ſo on to the laſt.

9. Ratio of equality, is that which equal quantities bear to each other.

It may be obſerved here, that ratio of equality, and equality of ratios, are, by no means, ſynonymous terms : ſince

ſince two or more ratios may be equal, though the quantities compared are all unequal. Thus, the ratio of 2 *to* 1, *is equal to the ratio of* 6 *to* 3, (2 *being the double of* 1, *and* 6 *the double of* 3); *yet none of the four numbers are equal.*

10. Inverſe ratio is, when the antecedent is made the conſequent, and the conſequent the antecedent. *Thus, if* 2 : 1 : : 6 : 3; *then, inverſly,* 1 : 2 : : 3 : 6.

11. Alternate proportion is, when antecedent is compared with antecedent, and conſequent with conſequent.

As, if 2 : 1 : : 6 : 3; *then,* by alternation (or permutation) *it will be* 2 : 6 : : 1 : 3.

12. Compounded ratio is, when the antecedent and conſequent, taken as one quantity, are compared either with the conſequent, or with the antecedent.

Thus, if 2 : 1 : : 6 : 3; *then,* by compoſition, 2 + 1 : 1 : : 6 + 3 : 3, *and* 2 + 1 : 2 : : 6 + 3 : 6.

13. Divided ratio is, when the difference of the antecedent and conſequent is compared, either with the conſequent, or with the antecedent.

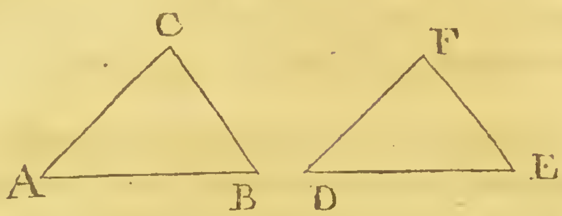
Thus, if 3 : 1 : : 12 : 4; *then* by diviſion, 3 — 1 : 1 : : 12 — 4 : 4, and 3 — 1 : 3 : : 12 — 4 : 12.

Theſe four laſt definitions, which explain the names given by Geometers *to the different ways of managing and diverſifying of proportions, are put down here for the ſake of order; but are not to be uſed, or referred to, in any ſhape, till thoſe properties and relations are demonſtrated; which is effected in the three firſt Theorems of this book.*

14. Similar (or like) right lined figures are ſuch, which have all their angles equal, one to another

 reſpec-

reſpectively, and alſo the ſides about the equal angles proportional.



Thus, if the angle A = D, B = E, C = F; alſo AC : AB : : DF : DE, BA : BC : : ED : EF, *&c.* then the figures ABC, DEF are ſaid to be *ſimilar.*

AXIOMS.

1. The ſame quantity being compared with ever ſo many equal quantities, ſucceſſively will have the ſame ratio to them all.

2. Equal quantities, have to one and the ſame quantity, the ſame ratio.

3. Quantities having the ſame ratio to one and the ſame quantity, or to equal quantities, are equal among themſelves.

4. Quantities, to which one and the ſame quantity has the ſame ratio, are equal.

5. If two quantities be referred to a third, that which is the greateſt will have the greateſt ratio.

6. If two quantities be referred to a third, that is the greateſt which has the greateſt ratio.

7. Ratios, equal to one and the ſame ratio, are alſo equal, one to the other.

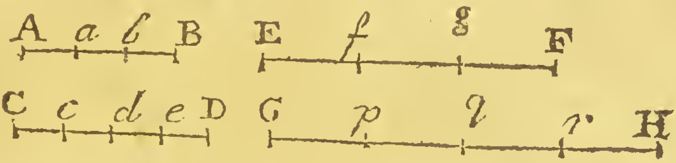
8. If two quantities be divided into, or compoſed of parts, that are equal among themſelves, or all of the ſame magnitude; then will the whole of the one, have the ſame ratio to the whole of the other, as the number of the parts in the one, has to the number of equal parts in the other.

9. If the double, treble, or quadruple, *&c.* of every part of any quantity be taken, the aggregate will

will be the double, treble, or quadruple, *&c.* of the whole quantity propounded.

THEOREM I.

Equimultiples of any two quantities (AB, CD) *are in the ſame ratio as the quantities themſelves.*



Let the ratio of AB to CD be that of any one number *M* (3) to any other number *N* (4), or, which is the ſame, let AB contain *M* (3) ſuch equal parts (A*a*, *ab*, *b*B [a]), whereof CD contains the num- [a] Ax. 8.4.
ber *N* (4). Let there be taken E*f*, *fg*, *g*F any equimultiples of A*a*, *ab*, *b*B, reſpectively; and let G*p*, *pq*, *qr*, *r*H be the ſame multiples of C*c*, *cd*, *de*, *e*D: ſo ſhall the whole EF be the ſame aſſigned multiple of the whole AB, and the whole GH of the whole CD, as each part in the one, is of its correſpondent in the other [b]. And, ſince the parts A*a* *ab*, [b] Ax. 9.4.
*b*B, C*c*, *cd*, &c. are all equal [c], their equimultiples [c] Hyp.
E*f*, *fg*, *g*F, G*p*, *pq*, &c. will alſo be equal [d]. There- [d] Ax. 4.1.
fore EF is in proportion to GH, as the number of the parts in EF is to the number of equal parts in GH [a], or (which is the ſame) as the number of parts in AB to the number of parts in CD, that is, as AB is to CD [a]: which was to be demonſtrated.

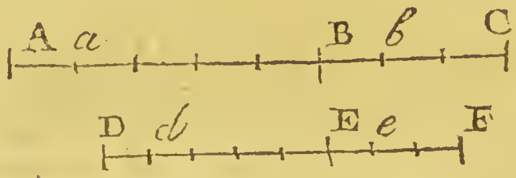
COROLLARY.

Hence, like parts of quantities, have the ſame ratio as the wholes; becauſe the wholes are equimultiples of the like parts.

 THEO-

THEOREM II.

The two antecedents (AB, DE) *of four proportional quantities, of the same kind* (AB, BC, DE, EF) *are in the same ratio with the two consequents* (BC, EF).



Let the common ratio of AB to BC, and of
DE to EF, be that of any one number *M* (5) to
any other number *N* (3); then will AB contain *M*
(5) such equal parts (A*a*) whereof BC contains
[e] Ax. 8. of 4. [e] *N* (3); and DE will, in like manner, contain *M*
(5) such equal parts (D*d*), whereof EF contains
N (3). And so, AB and DE, as well as BC and EF,
being equimultiples of A*a* and D*d*, thence will AB
[f] 1 4. : DF : : A*a* (B*b*) : D*d* (E*e*) : : BC : EF [f].

COROLLARY.

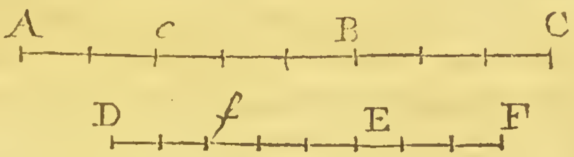
That the proportionality will subsist, when the consequents are taken as antecedents, and the antecedents as consequents, also appears from hence. For BC : AB : : number of parts in BC (or EF) : number of parts in AB (or DE) : : EF : DE [e].

THEOREM III.

*Of four proportional quantities (*AB, BC, DE, EF*) the sum, or difference of the first antecedent and consequent* (AB ± BC) *is to the first antecedent, or consequent, as the sum, or difference of the second antecedent and consequent* (DE ± EF) *is to the second antecedent, or consequent.*

Let

Let what was premised in the demonstration of the preceding theorem, be retained here: then will



AC (AB + BC) be in proportion to AB, as the number of parts in AC is to the number of equal parts in AB [g], or as the number of parts in DF [g] Ax.8.4. (DE + EF) to the number of equal parts in DE, that is, as DF (DE + EF) is to DE [g]. Again, if from AB and DE, be taken away B*c* = BC, and E*f* = EF, then will the difference A*c* be in proportion to AB, as the number of parts in A*c* (or D*f*) is to the number of parts in AB (or DE [g]), that is, as D*f* is to DE. In the same manner it will appear, that AB + BC : BC : : DE + EF : EF; and AB — BC : BC : : DE — EF : EF.

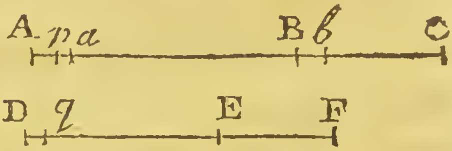
COROLLARY.

It will appear from hence, that the sum of the greatest and least (AB + EF) of four proportional quantities (of the same kind) will exceed the sum (BC + DE) of the two means: because, AB being supposed greater than DE, A*c* will be greater than D*f*, in the same proportion [h]: and, if to these [h] 3. 4. there be added BC + EF (common;) then will the sum A*c* + BC + EF (AB + EF) be also greater than the sum D*f* + BC + EF [i] (DE + BC.) [i] Ax. 6.1.

SCHOLIUM.

In the demonstration of this, and the preceding theorems, the antecedent and consequent are supposed to be divided into parts, all mutually equal among themselves. But it is known to Mathematicians, that there are certain quantities, or mag-

nitudes that cannot poſſibly be divided in that manner, by means of a *common meaſure.* The Theorems themſelves are, nevertheleſs, true, when applied to theſe *incommenſurables:* ſince no two quantities, of the ſame kind, can poſſibly be aſſigned, whoſe ratio cannot be expreſſed by that of two numbers, ſo near, that the difference ſhall be leſs than the leaſt thing that can be named. But if the matter, viewed in this light, ſhould not appear ſufficiently *ſcientific*, and you will not (in the preceding theorem) allow the ratio of AC to BC, to be *exactly* the ſame with that of DF to EF, when AB, BC, and DF, EF, are incommeaſurables; then let it, if poſſible, be as ſome quantity aC (leſs than AC) is to BC, ſo is DE to EF. Let Bb be a part or meaſure of BC leſs than the difference (Aa) between AB and aB; let Bp be that multiple of Bb, which leaſt exceeds Ba, and let qE be to EF, as pB to BC.



It is evident, that pB is leſs than AB (becauſe $ap \sqsupset$ B$b \sqsupset$ [k] Aa); and that qE is alſo leſs than DE, because the ratio of qE to EF, being equal to that of pB to BC [k], it muſt neceſſarily be leſs than that of AB to BC [l], or of DE to EF, and ſo qE leſs than DE [m].

[k] Hyp.
[l] Ax. 5.4.
[m] Ax. 6.4.

Now, if (as is ſuppoſed) the ratio of aC to BC can be = the ratio of DF to EF, it muſt, of conſequence, be greater than the ratio of qF to EF [n], or (which is the [o] ſame) than the ratio of pC to BC [l], *which is impoſſible.* In like manner it will appear, that no quantity, that is greater than AC, can poſſibly be to BC, as DF is to EF. Therefore AC:BC :: DF : EF. And by the ſame kind of argumentation (authorized and adopted by *Euclid* himſelf, in

[n] Ax. 5. and 7.4
[o] Proof of 3. 4.

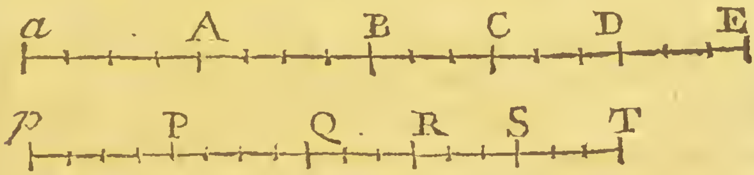
in his twelfth book) any difficulties, or ſcruples, that may be elſewhere brought, from the incommenſurability of quantities, may be obviated and removed.

THEOREM IV.

If, of four proportionals (AB, BC, PQ, QR) *equimultiples of the antecedents* (AB, BQ) *be taken, and compared with any equimultiples of their reſpective conſequents* (BC, QR), *the ratios will be the ſame, and the four quantities proportionals.*

Let the common ratio of AB to BC, and of PQ to QR, be that of any one number M to any other number N: ſo ſhall AB contain M ſuch equal parts whereof BC contains [p] N, and PQ, in like manner, M ſuch equal parts whereof QR contains N. [p] Ax.8.4.

Let CD, DE be taken each = BC, and RS, ST each = QR, ſo that BE and QT may be equimultiples of BC and QR; and let CD, DE; RS, ST be conceived to be divided, each into the ſame number of parts with BC, or QR. In like man-

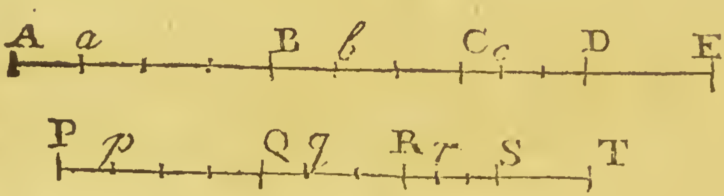


ner, let aB and pQ be taken as equimultiples of AB and PQ, &c. Then will the number of parts in BD = number of parts in QS [p], and the number of parts in BE = number of parts in QT [q]: [q] Ax.4.1.
and ſo likewiſe with reſpect to aB and pQ. Therefore aB is to BE, as the number of parts in aB to the number of (equal) parts in BE [p], or, which is the ſame thing, as the number of parts in pQ to the number of parts in QT, that is, as pQ is to QT, *which was to be demonſtrated.*

THEOREM V.

If of two ranks of quantities (AB, BC, CD; PQ, QR, RS), *the ratio of the firſt and ſecond, in the one, be equal to the ratio of the firſt and ſecond in the other, and the ratio of the ſecond and third, in the one, equal likewiſe to the ratio of the ſecond and third in the other; then, alſo, ſhall the ratio of the firſt to the third, be the ſame in the one rank, as in the other.*

Let the common ratio of AB to BC, and of PQ to QR, be ſtill expreſſed as in the preceding demonſtrations. Let, moreover, CD and RS be conceived to be divided, each into the ſame number of parts with BC and QR.



Becauſe the quantities BC, CD, QR, RS,
[r] Hyp. are proportional [r], their like parts Bb, Cc, Qq, Rr
[s] Cor. to 1. 4. (being in the ſame ratio with the wholes [s]) will alſo be proportionals; or, becauſe Aa = Bb and Pp =
[t] Ax. 2. 4. Qq [r], it will be [t] Aa : Cc : : Pp : Rr. But AB and PQ are *equimultiples* of the antecedents Aa and Pp [r]; and CD, RS are *equimultiples* of the conſequents
[u] 4. 4. Cc and Rr; therefore [u] AB : CD : : PQ : RS *; *which was to be demonſtrated.*

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* *When in two ranks of quantities, the proportions are* inordinate, *as* AB : BC : : QR : RS, and BC : CD : : PQ : QR; *the ſame thing may be demonſtrated; and that in the very ſame manner, except only, that* QR *muſt here be divided into the ſame number of parts with* AB, *and* PQ.

COROL-

COROLLARY I.

If other quantities DE, ST be taken, ſtill proportional to the two next preceding them, ſo that CD : DE : : RS : ST; then, by the ſame argument (regard being had to AB, CD, DE in the one rank, and PQ, RS, ST in the other) it is evident, that AB : DE : : PQ : ST [x]. And thus we may go on, ſtill aſſuming other quantities, as many as we pleaſe; and the ratio of the firſt and laſt, will always be the ſame in one rank, as in the other. Therefore ratios [y] compounded of the ſame number of like, or equal ratios, are equal.

[x] 5. 4.
[y] Def. 8. of 4.

COROLLARY II.

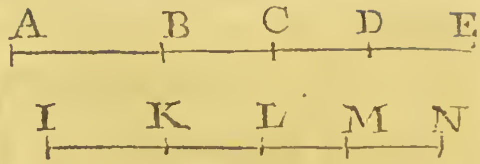
It is alſo evident from hence, that if any two quantities be taken proportional to the two conſequents of an aſſigned proportion, they will alſo be proportionals when compared with antecedents; and *vice verſâ*. For, the two quantities CD and RS, when compared, ſucceſſively, with the conſequents, and antecedents of the given proportion AB : BC : : PQ : QR, appear to be proportional, in the one caſe, as well as in the other [z].

[z] 5. 4.

THEOREM VI.

If, to the two conſequents (BC, KL) *of four proportionals* (AB, BC, IK, KL), *any two quantities* (CD, LM) *that have the ſame ratio to the reſpective antecedents be added; theſe ſums and the antecedents will ſtill be proportionals (I ſay, if* AB : BC : : IK : KL, *and* AB : CD : : IK : LM; *then ſhall* AB : BD : : IK : KM.)

For,



For, CD and LM being proportional to the antecedents
[a] Hyp. AB and IK[a], they and the conſequents (BC, KL) will alſo be proportionals (*by Corol.* 2.
[b] 3. 4. *of the precedent*) : whence [b] (*by compoſition*) BC : BD : : KL : KM. And ſo again, (*by the ſame Corol.*) AB : BD : : IK : KM.

COROLLARY.

From this Theorem it will appear, that, if the ratios of the correſponding quantities of two ranks with reſpect to the two firſt, are the ſame in both ranks (AB : BC : : IK : KL, AB : CD : : IK : LM, *&c.*); then the ratio of all the quantities to the firſt, will alſo be the ſame in the one rank, as in the other. For, by adding DE and MN to the laſt
[c] 6. 4. conſequents (BD, KM) there will be had [c] AB : BE : : IK : KN (and ſo on, to any number of quantities whatever.) Then (by compoſition) AB : AE : : IK : IN.

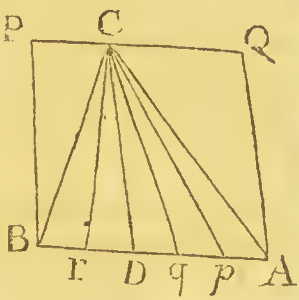
When the quantities in both ranks, are of the ſame kind, it will appear (by alternation) that the ratio of the two ſums, and *that* of every two correſponding terms, will be the ſame.

The ſix Theorems here delivered, on proportions of magnitudes in general, comprehend all that is moſt uſeful in that ſubject.—What relates to the proportions of extended magnitudes, under different limitations, and figures, as far as regards right lines and ſurfaces, will be the ſubject of the remaining part of this book.

Note, when-ever, in any demonstration, you meet with several proportional quantities, connected continually by the sign : : (*like these*, A : B : : C : D : : E : F : : G : H) *the conclusion to be drawn, is always from the first and last of the two equal ratios.*

THEOREM VII.

Triangles (ACD, BCD), *and also parallelograms* (ADCQ, BDCP), *having the same altitude, are to one another in the same ratio as their bases* (AD, BD).



Let the base AD be to the base BD in the ratio of any one number m (3) to any other number n (2), or which is the same[c], let AD contain m (3) such equal parts whereof BD contains the number n (2). Then, the triangles ACp, pCq, BCr, &c. made by drawing lines from the points of division to the vertex C, being all equal among themselves[d]; the triangle ACD will therefore be in proportion to the triangle BCD, as the number of equal parts in the former to the number of equal parts in the latter, or as the number of parts in AD to the number of parts in BD, that is, as AD to BD[c], whence, also, the parallelograms ADCQ, BDCP, being the doubles of their respective triangles[d], are likewise in the same ratio as their bases AD and BD[e].

[c] Ax. 8.4.

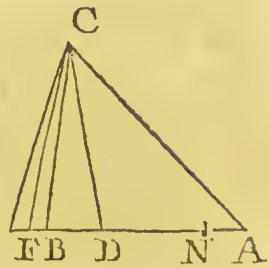
[d] Cor. 2. to 2. 2.

[e] 4. 4.

SCHOLIUM.

If the bases AD *and* BD *are incommensurable to each other, the ratio of the triangles cannot be other than that of their bases.*

For,



For, if poſſible, let the triangle BCD be to the triangle ACD, not as BD to AD, but as ſome other line ED, greater than BD, is to AD.

[e] Lem. 1. 8.

Let AN be a part, or meaſure of AD, leſs than BE [e], and let DF be that multiple of AN which leaſt exceeds DB; alſo let CE and CF be drawn. It is manifeſt that the point F falls between B and E, becauſe (*by Hyp.*) BF is leſs than AN, and AN leſs than BE. Moreover, the ratio of FCD to ACD is the ſame as that of FD to AD (*by the precedent.*) But the ratio of BCD to ACD (or of ED to AD [f]) is ⊏ the ratio of FD to AD [g], or of FCD to ACD; and conſequently BCD ⊏ FCD [h]; *which is impoſſible* [i] by the ſame argument it will appear, that the triangle BCD cannot be to the triangle ACD, as a line, leſs than BD is to AD. Therefore BCD : ACD : : BD : AD.

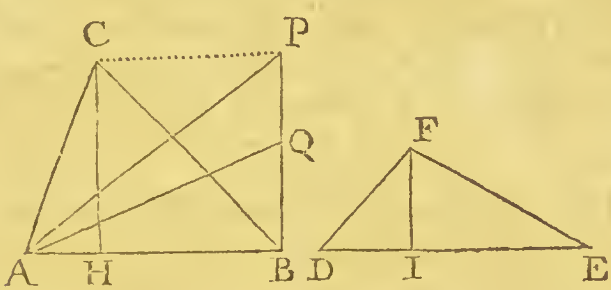
[f] Hyp.
[g] Ax. 5. of 4.
[h] Ax. 6. of 4.
[i] Ax. 2. of 1.

If this Scholium ſhould appear difficult to the Learner, it may not be amiſs to omit it intirely; ſince it is only put down for the ſake of thoſe who may be ſcrupulous about the buſineſs of incommenſurables; to whom it may not be improper to obſerve, that nothing more is taken for granted herein, than what is effected by means of the firſt Lemma in the 8th book; which being demonſtrated from axioms, and one ſingle theorem in the firſt book, is referred to here, though not given till hereafter, for reaſons already hinted at, in this note.

THEO-

THEOREM VIII.

Triangles (ABC, DEF) *standing upon equal bases* (AB, DE) *are to one another, in the same ratio as their altitudes* (CH, FI.)



Let BP be perpendicular to AB, and equal to CH; in which let there be taken BQ = FI, and let AP and AQ be drawn.

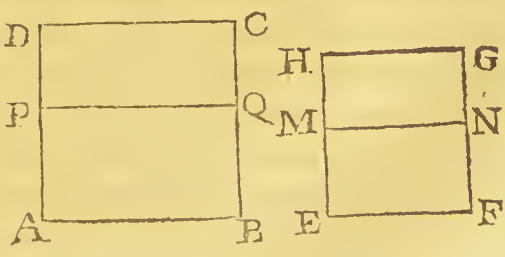
The triangle ABP is = ABC[k], and ABQ = DEF[k]; but ABP (ABC) : ABQ (DEF) : : [l] BP (HC) : BQ (FI).

[k] Cor. 2. to 2. 2.
[l] 7. 4.

THEOREM IX.

If, parallel to the bases of any two parallelograms (AC, EG), *two lines* (PQ, MN) *be drawn, so as to cut the sides proportionally* (AP : AD : : EM : EH), *then will those parallelograms and their corresponding parts* (AQ, EN) *be also proportionals.*

For, AQ : AC : : AP : AD[m] : : EM : EH[n] : : EN : EG:[m]; and therefore, *by alternation*, AQ : EN : : AC : EG[o].



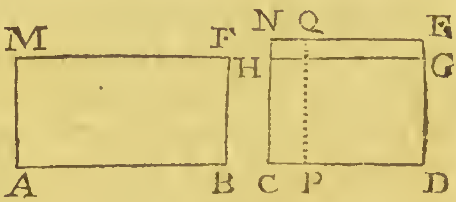
[m] 7. 4.
[n] Hyp.
[o] 2. 4.

THEO-

THEOREM X.

If four lines are proportional (AB : CD : : DE : BF), *the rectangle* (AF) *under the two extremes will be equal to the rectangle* (CE) *under the two means. And, if the rectangles under the extremes and means of four given lines* (AB, CD, DE, BF) *be equal, then are thoſe four lines proportional.*

In DE let DG be taken = BF, and let GH, parallel to DC, be drawn.



1. *Hyp.* AF : CG
[p] 7. 4. : : AB : CD[p] : : DE
[q] Hyp. : BF[q] : : DE : DG[r]
[r] Ax. 1. 4. : : CE : CG[p]; there-
fore, the conſequents of the firſt and laſt of theſe equal ratios being the ſame quantity CG,
[s] Ax. 3. the two antecedents AF and CE muſt be equal[s].
[t] Ax. 2. 4. 2. *Hyp.* AB : CD : : AF : CG[p] : : CE : CG[t] : : DE : DG[p] (BF.)

SCHOLIUM.

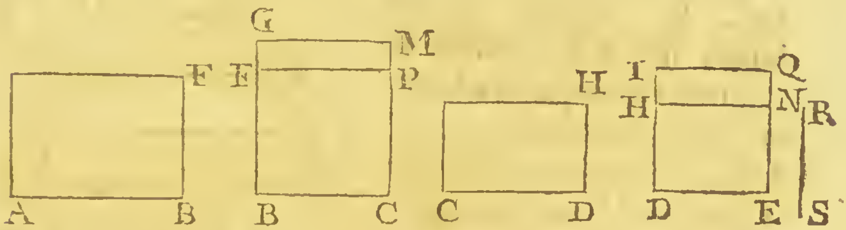
From the ſame demonſtration, and ſcheme, it will appear, that the two antecedents of four proportional lines (AB, CD, DE, BF) are in the ſame ratio to each other, as the two conſequents: for, if in DC there be taken DP = BF, and PQ be
[t] 7. 4. drawn parallel to DE; then AB : DE : : AF : PE[t] : : CE : PE : : CD : PD (PF).

THEO-

THEOREM XI.

The rectangles under the corresponding lines, of two ranks of proportionals, are themselves proportionals. (*I say, if* AB : BC : : CD : DE, *and* BF : BG : : DH : DI, *then will the rectang.* AF : *rectang.* BM : : CH : *rect.* DQ).

For, in BG and DI (produced if neceſſary) let there be taken BF = BF, DH = DH, and let



FP be parallel to BC, and HN to DE : then AF : BP : : AB : BC [y] : : CD : DE [z] : : CH : DN [y] ; [y] 7. 4.
whence (*alternately*) AF : CH : : PB : DN, and [z] Hyp.
ſo likewiſe is BM to DQ [a] : whence (*again by alternation*) AF : BM : : CH : DQ. [a] 9. 4.

COROLLARY I.

Hence, the ſquares of four proportional lines, are themſelves proportional.

COROLLARY II.

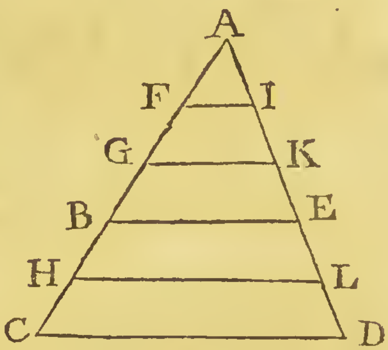
Hence alſo, the ſides of four proportional ſquares (AB^2, BC^2, CD^2, DE^2) will be proportional. For, let the line RS be taken ſuch, that, AB : BC : : CD : RS ; then, ſince $AB^2 : BC^2 : : CD^2 : RS^2$ (*by Corol.* 1.) and $AB^2 : BC^2 : : CD^2 : DE^2$ *by ſuppoſition*) : thence will [b] $RS^2 = DE^2$; therefore [c] RS = DE, and conſequently AB : BC : : CD : DE [d].

[b] Ax. 4. 4.
[c] Cor. 2. of 6. 2.
[d] Ax. 1. 4.

THEO-

THEOREM XII.

A line (BE) *drawn parallel to one side* (CD) *of a triangle* (ACD) *divides the other two sides proportionally.* (I say, AB : AC : : AE : AD, AB : BC : : AE : ED, *and* AC : BC : : AD : ED).



Let AB be to AC, as any one number *m* (3) is to any other number *n* (5), or, which is the same, let AB contain *m* (3) such equal parts whereof AC contains *n* (5). Then, if from the points of division, lines be drawn parallel to the side CD, they will also divide AE and AD into the like numbers of equal parts [e], therefore AE is to AD, as the number of equal parts in AE to the number of equal parts in AD, or as the number of equal parts in AB to the number of equal parts of AC, that is, as AB to AC [f]: In the same manner, AE is to ED, as the number of parts in AE to the number of parts in ED, or as the number of parts in AB to the number of parts in BC, that is, as AB to BC. Also, in the same manner, AC : BC : : AD : ED.

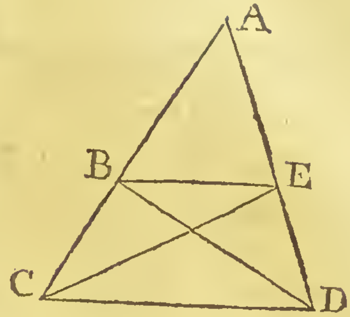
[e] Cor. 1. to 27. 1.
[f] Ax. 8.

The same otherwise.

Draw CE and DB. Then will the triangles BEC and EBD be equal to each other [f]; whence, by adding BEA to both, AEC will be also = ABD [g].

[f] Cor. 1. to 2. 2.
[g] Ax. 4. 1.

But,



But, AB : AC : : triang. AEB : triangle AEC [h] (ABD) : : AE : AD [h]; and AB : BC : : triang. AEB : triangle CEB [h] (DEB) : : AE : ED.

[h] 7. 4.

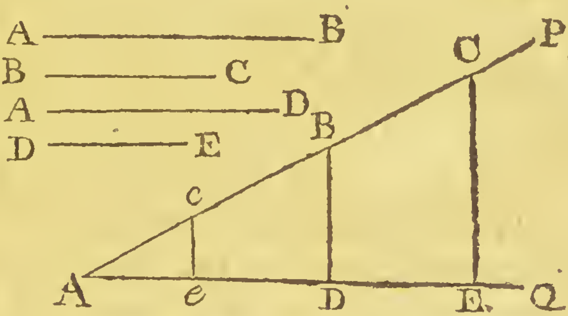
COROLLARY.

Hence a right-line, which divides two ſides of a triangle proportionally, is parallel to the remaining ſide: becauſe AD is divided in the ſame ratio with AC, when BE is parallel to CD; but not elſe [i].

[i] Ax. 2. 1. and 5. 4.

SCHOLIUM.

From this laſt Theorem, whatever relates to the compoſition and diviſion of ratios, when theſe reſpect the compariſon of right-lines will appear exceedingly obvious.



For, let AB, BC, AD, and DE, be proportionals: and from any point A, let two indefinite right-lines AP, AQ be drawn; in which take AB = AB, BC = BC, AD = AD, and DE = DE; alſo take B*c* = BC, D*e* = DE, and let BD, CE, and *ce* be drawn.

Since AB : BC : : AD : DE [k], thence is EC parallel to DB [l]; and ſo, B*c* being = BC, and D*e* = DE, *ec* will alſo be parallel to DB [m]. Therefore,

AC (AB+BC) : AB : : AE (AD+DE) : AD;
AC (AB+BC) : BC : : AE (AD+DE) : DE;

[k] Hyp.
[l] Cor. of 12. 4.
[m] Cor. 2. to 27. 1.

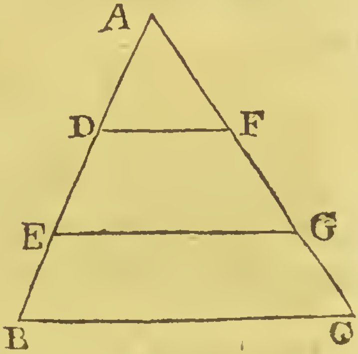
Ac (AB — BC) : AB : : *Ae* (AD — DE) : AD ;

Ac (AB — BC) : BC : : *Ae* (AD — DE) : DE ;

And, AC (AB + BC) : *Ac* (AB — BC) : : AE (AD + DE) : *Ae* (AD — DE).

THEOREM XIII.

The parts (DE, FG) *of the two ſides of a triangle, intercepted by right-lines* (DF, EG) *drawn parallel to the baſe* (BC), *are in the ſame ratio with the wholes.* (DE : FG : : AB : AC).



For, DF and EG being parallel to each other, thence will DE
o 12. 4. : AE : : FG : AG [o]; therefore, by (alternation) DE : FG : : AE :
p 2. 4. AG [p]. In the ſame manner, AE : AG : : AB : AC [o]. Conſequently DE
q Ax. 7. 4. : FG : : AB : AC [q].

COROLLARY.

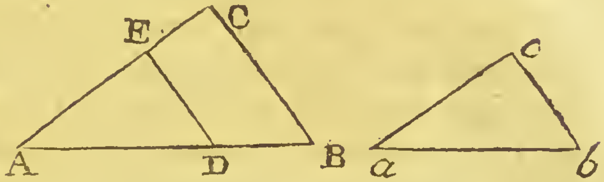
Hence, if ever ſo many lines be drawn parallel to the baſe, cutting the ſides of a triangle, every two correſponding ſegments will have the ſame ratio [q].

THEOREM XIV.

In triangles (ABC, abc) *mutually equiangular the correſponding ſides* (AB, ab, AC, ac) *containing the equal angles* (A, a) *are proportional.*

In

In AB and AC (produced if neceſſary) tak AD $=$ *ab*, and AE $=$ *ac*, and join D, E.



The triangles *abc* and ADE, having *ab* $=$ AD, *ac* $=$ AE, and the angle *a* $=$ A, have alſo the angle ADE[r] $=$ *abc* $=$ ABC[s]; whence DE will be parallel to BC[t]; and therefore AB : AD (*ab*) : : AC : AE[u] (*ac*).

[r] Ax. 10. 1.
[s] Hyp.
[t] Cor. to 8. 1.
[u] 12. 4.

The ſame otherwiſe.

Becauſe AB $\times$ AC $=$ *ab* $\times$ AC[w], therefore is AB : *ab* : : AC : *ac*[x].

[w] 24. 3.
[x] 10. 4.

COROLLARY.

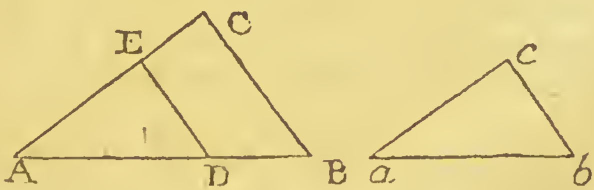
Hence equiangular triangles are ſimilar to each other[y].

[y] Def. 14. of 4.

THEOREM XV.

If two triangles (ABC, abc) *have one angle* (BAC) *in the one, equal to one angle* (bac) *in the other, and the ſides* (AB, ab, AC, ac) *about thoſe angles proportional; then are the triangles equiangular.*

In AB and AC, take AD $=$ *ab*, and AE $=$ *ac*, and let DE be drawn.



Since AB : *ab* (AD) : : AC : *ac* (AE[b]), therefore is DE parallel to BC[c]; whence the angle B $=$ ADE[d] $=$ *b*[e], and the angle C $=$ AED $=$ *c*.

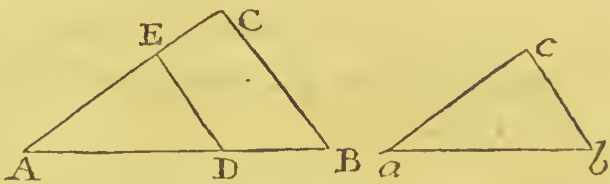
[b] Hyp.
[c] Cor. to 12. 4.
[d] Cor. 1. to 7. 1.
[e] Ax. 10. 1.

 THEO-

THEOREM XVI.

If two triangles (ABC, abc) *have one angle* (A) *in the one, equal to one angle* (a) *in the other, and the ſides* (AB, ab, CB, cb) *about either of the other angles proportional; then will the triangles be equiangular, provided theſe laſt angles* (B, b) *be, either, both leſs, or both greater, than right-angles.*

In AB, let AD be taken = *ab*, and let DE be drawn parallel to BC, meeting AC in E.



Then will the triangles ABC, and ADE, be equiangular[f]; therefore, CB : ED : : AB : AD[g] : : AB : *ab*[h] : : CB : *cb*[i]; and conſequently ED = *cb*[k]: whence the triangles *abc* and ADE (having *ab* = AD, *cb* = ED, and *a* = A) will be equal in all reſpects[l], provided the angles *abc* and ABC (= ADE) are either both leſs, or both greater than right angles. Therefore, ſince the latter of theſe equal triangles (*abc*, ADE) is equiangular to ABC, the propoſition is manifeſt.

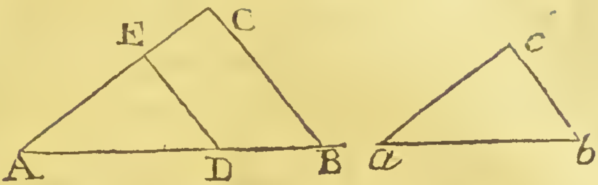
f Cor. 1. to 7. 1.
g 14. 4.
h Ax. 1 4.
i Hyp.
k Ax. 4. 4.
l 17. 1.

THEOREM XVII.

If two triangles (ABC, abc) *have all their ſides, reſpectively proportional* (AC : ac : : AB : ab : : CB : cb) *then are thoſe triangles equiangular.*

In AC and AB, take AE = *ac*, and AD = *ab*, and join E, D.

Since

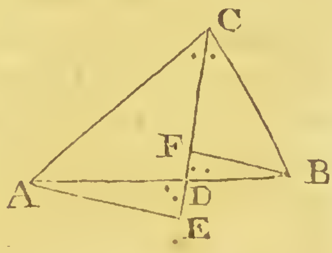


Since AC : AE (*ac*) : : AB : AD (*ab*) [m], the triangles ABC, ADE are equiangular [n]; hence CB : ED : : [n] AB : AD [o] (*ab*) : : CB : *cb* [m]; and consequently [o] ED = *cb* [p]: therefore the triangles *abc*, ADE, [p] being mutually equilateral, they must also be mutually equiangular [q]; and consequently *abc*, as well [q] as ADE, equiangular to ABC.

[m] Hyp.
[n] 15. 4.
[o] 14. 4.
[p] Ax. 4. 4.
[q] 14. 1.

THEOREM XVIII.

A right-line (CD) *bisecting any angle* (ACB) *of a triangle* (ABC) *divides the opposite side* (AB) *into two segments* (AD, BD), *having the same ratio with the sides* (AC, CB) *containing that angle.*

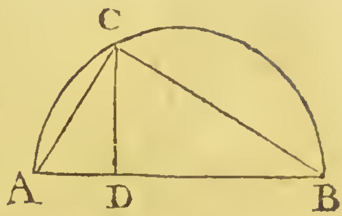


Let AE and BF be perpendicular to CDE. Then the triangles ACE, CBF, and ADE, BDF being, respectively, equiangular [r], it will be AD : BD : : AE : BF [s] : : AC : BC.

[r] Hyp. and 3. 1.
[s] 14. 4. and 2. 4.

THEOREM XIX.

A *perpendicular* (CD) *let fall from the right-angle* (C) *upon the hypothenuse* (AB) *of a right-angled triangle* (ABC) *will be a mean proportional between the two segments* (AD, BD) *of the hypothenuse: and each of the sides containing the right angle, will be a mean proportional between its adjacent segment, and the whole hypothenuse.*



[t] Ax. 7. For ſince the angle BDC is = BCA [t], and B common, the triangles BDC, BCA are equiangular [u]: after the ſame manner ADC and ABC appear to be equiangular.

[u] Cor. 1. to 10. 1.

Therefore, by Theor. XIV.

BD : CD : : CD : AD
AB : BC : : BC : BD
AB : AC : : AC : AD.

COROLLARY.

Becauſe the angle in a ſemi-circle is a right-angle [w], it follows, that, if from any point C, in the periphery of a ſemi-circle ACB, a perpendicular CD be let fall upon the diameter AB, and from the ſame point C, to the extremities of that diameter, two chords CA, CB be drawn; the ſquare of that perpendicular will be equal to a rectangle under the two ſegments of the diameter; and the ſquare of each chord, equal to a rectangle under the whole diameter and its adjacent ſegment: for, becauſe of the above proportions, we have $CD^2 = BD \times AD$, $BC^2 = AB \times BD$, and $AC^2 =$ [x] $AB \times AD$.

[w] 13. 3.

[x] 10. 4.

THEOREM XX.

If, in ſimilar triangles (ABC, EFG) *from any two equal angles* (ACB, EGF) *to the oppoſite ſides, two right lines* (CD, GH) *be drawn, making equal angles with the homologous ſides* (CB, GF); *thoſe right-lines will have the ſame ratio as the ſides* (AB, EF) *on which they fall, and will alſo divide thoſe ſides proportionally.*

For,

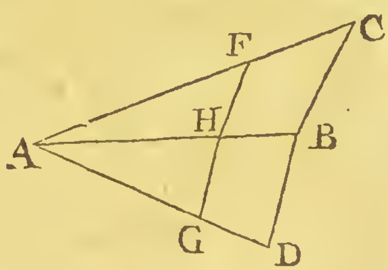
For, the triangles ADC, EHG, and BDC, FHG (as well as the *wholes* ABC, EFG) being equiangular[y].
thence is[z] AB : EF (: : AC : EG) : : CD : GH; and AD : EH (: : DC : HG) : : BD : FH.

[y] Hyp.
[z] 14. 4. and Ax. 7.

THEOREM XXI.

If in two triangles (ABC, ABD) *having one side* (AB) *common to both, from any point* H *in that side, two lines* (HF, HG) *respectively parallel to two contiguous sides* (BC, BD) *be drawn, to terminate in the two remaining sides* (AC, AD); *those lines* (HF, HG) *will have the same ratio as the sides* (BC, BD) *to which they are parallel.*

For, AB : AH : : BC : HF[a], and AB : AH : : BD : HG[a]; therefore, by equality, BC : HF : : BD : HG; whence, alternately, BC : BD : : HF, HG[b].



[a] 14. 4.
[b] 2. 4.

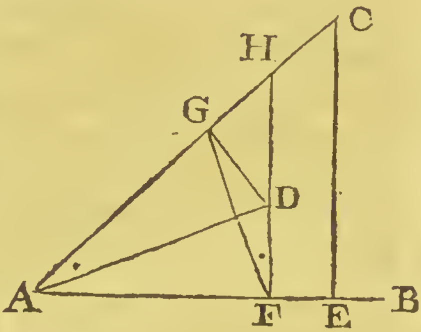
COROLLARY.

Hence, if BC = BD, then also will HF = HG.

THEOREM XXII.

If, at any two points (F, G) *in two lines* (AB, AC) *meeting each other, two perpendiculars* (FD, GD) *be erected, ſo as to meet each other; the diſtance* (AD) *of their concourſe from that of the propoſed lines, will be to the diſtance* (FG) *of the two points themſelves, in the ratio of one of the ſaid lines* (AC) *to a perpendicular* (CE) *falling from the extreme thereof upon the other* (AB).

Let FD be produced to meet AC in H.



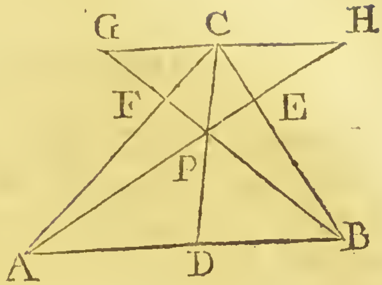
Since the angles AFD and AGD are right-ones[c], the circumference of a circle will paſs thro' all the four points A, F, D, G[d]; and ſo the angles GFD, GAD, ſtanding on the ſame ſubtenſe GD, will be equal[e]; and conſequently the triangles AHD, FHG equiangular[f]: therefore AD : FG : : AH : HF[g] : : AC : CE.

[c] Hyp.
[d] 19. 3.
[e] 11. 3.
[f] Cor. 1. to 10. 1.
[g] 14. 4.

THEOREM XXIII.

If thro' any point (P) *in a triangle* (ABC) *three right-lines* (AE, BF, CD) *be drawn, from the angular points to cut the oppoſite ſides, the ſegments* (AD, BD) *of any one ſide* (AB) *will be to each other, as the rectangles* (AF × CE, BE × CF) *under the ſegments of the other ſides taken alternately.*

Let



Let GCH be parallel to AB, and let AE and BF be produced to meet it in H and G.

It is manifest, that the triangles FBA, FCG; EAB, EHC; APB, GPH, are equiangular[h]:

Therefore AF : CF : : AB : CG[i],
CE : BE : : CH : AB[i].

Whence AF × CE : CF × BE : : CH × AB : CG × AB[k] : : CH : CG[l]; but CH : CG : : AD : BD[m]; therefore, by equality, AF × CE : CF × BE : : AD : BD.

[h] 7. and 3. of 1.
[i] 14. 4.
[k] 11. 4.
[l] 7. 4.
[m] 20. 4.

COROLLARY.

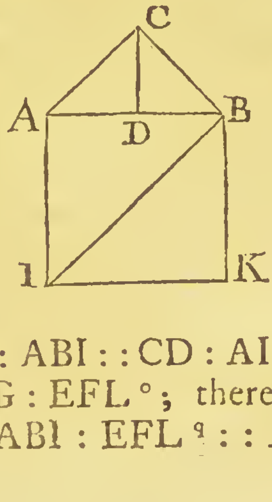
Hence, if AD = BD, then alfo will AF × CE = CF × BE, and therefore AF : CF : : BE : CE[n].

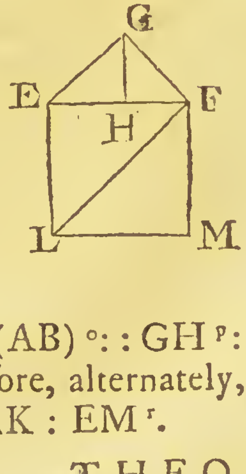
[n] 10. 4.

THEOREM XXIV.

Equiangular triangles (ABC, EFG) *are in proportion to one another, as the fquares* (AK, EM) *of their homologous fides.*

Upon AB and EF let fall the perpendiculars CD and GH, and let the diagonals BI, FL be drawn.





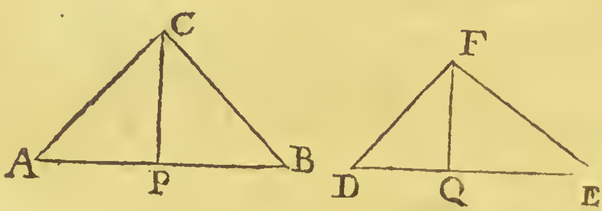
Becaufe ABC : ABI : : CD : AI (AB)[o] : : GH[p] : EF (EL) : : EFG : EFL[o]; therefore, alternately, ABC : EFG : : ABI : EFL[q] : : AK : EM[r].

[o] 8. 4.
[p] 20. 4.
[q] 2. 4.
[r] Cor. 2. to 2. 2.

THEO-

THEOREM XXV.

Triangles (ABC, DEF) *having one angle* (A) *in the one, equal to one angle* (D) *in the other, are in the ratio of the rectangles* (AC×AB, DF×DE) *contained under the sides including the equal angles.*



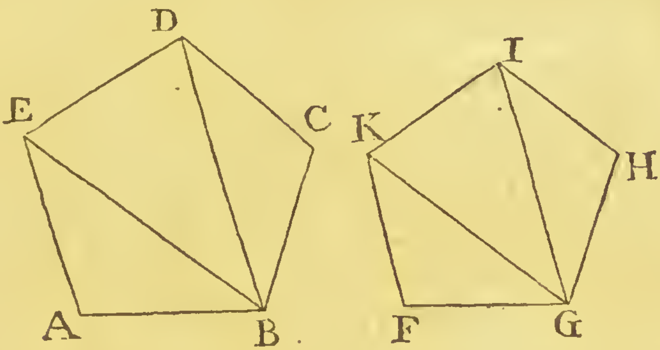
Upon AB and DE, let fall the perpendiculars CP and FQ. Then AC × AB : CP × AB : : AC :
[s] 7. 4. CP [s] : : DF : FQ [t] : : DF × DE : FQ × DE [s];
[t] 14. 4. whence, alternately, AC × AB : DF × DE : : CP
[u] Cor. 1. 4. × AB : FQ×DE : triangle ABC : : triangle DEF [u].

COROLLARY.

Hence, if the rectangles of the sides containing the equal angles, be equal, or the sides themselves
[w] 10. 4. reciprocally proportional [w], the triangles will be equal. The same also holds in parallelograms, being the doubles of such triangles.

THEOREM XXVI.

All similar right-lined figures (ABCDE, FGHIK) *are in proportion to one another as the squares of their homologous sides* (AB, FG).



Draw the right-lines BE, BD, GK, GI.

Because A=F, and AB : AE : : FG : FK [x], the triangles BAE, GFK are equiangular [y]; therefore, if from AED = FKI [x], there be taken AEB = FKG, the remainders BED, GKI will also be equal [z]. Wherefore, since ED : KI (: : EA : KF [x]) : : EB : KG [w], the triangles EBD, KGI are likewise equiangular [y]. In the same manner it will appear, that DBC, IGH are also equiangular.

[x] Def. 14. of 4.
[y] 15. 4.
[z] Ax. 5. of 1.
[w] 14. 4.

Therefore, because ABE : GFK (: : [a] BE² : GK²) : : BED : GKI (: : [a] BD² : GI²) : : [a] BDC : GIH, it is evident, that the sum of all the antecedents (ABCDE) is to the sum of all the consequents (FGHIK) as the first antecedent ABE is to the first consequent GFK [b], or as AB² to [a] FG²; *which was to be demonstrated.*

[a] 24. 4.
[b] Cor. to 6. 4.

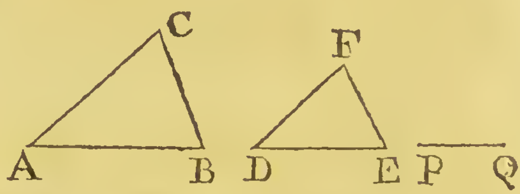
THEO-

THEOREM XXVII.

If three right-lines (AB, DE, PQ) *are proportional, the right-lined figure* (ABC) *upon the first, will be in proportion to the similar, and similarly described, figure* (DEF) *on the second, as the first line* (AB) *to the third* (PQ).

[c] 7. 4. [d] Hyp. and 10. of 4. [e] 26. 4.

For, AB : PQ : : AB^2 : AB × PQ [c] ([d] DE^2) : : ABC : DEF [e].



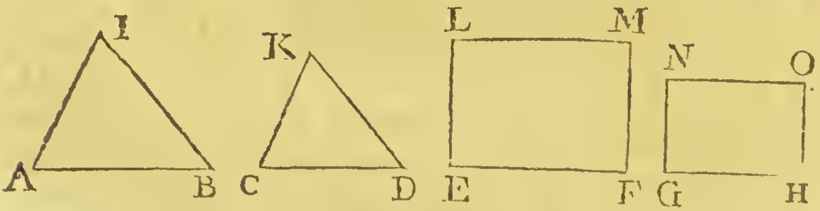
COROLLARY.

[f] Def. 7. of 4.

Hence, similar right-lined figures, are in the duplicate ratio of their homologous sides [f].

THEOREM XXVIII.

If four right-lines (AB, CD, EF, GH) *be proportional, the right-lined figures described upon them, being like, and in like sort situate, shall also be proportional* (ABI : CDK : : EM : GO).



[g] 26. 4. [h] Cor. 1. to 11. 4

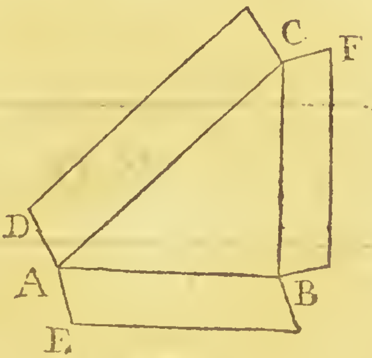
For, ABI : CDK (: : [g] $AB^2 : CD^2 : : EF^2 : GH^2$ [h]) : : EM : GO [g].

THEO-

THEOREM XXIX.

If upon the three ſides of a right-angled triangle (ABC) *as many right-lined figures* (CD, BE, BF) *like, and alike ſituate, be deſcribed, that* (CD) *upon the hypothenuſe* (AC) *will be equal to both the other two* (BE, BF) *taken together.*

For, BE : BF : : AB^2 : [l] BC^2; therefore (by compoſition) BE + BF : BE : : $AB^2 + BC^2$ (= [m] AC^2) : AB^2 : : CD : BE [l]; and conſequently BE + BF = CD [n].



[l] 26. 4.

[m] 8. 2.

[n] Ax. 3. of 4.

The End of the FOURTH BOOK.

ELEMENTS

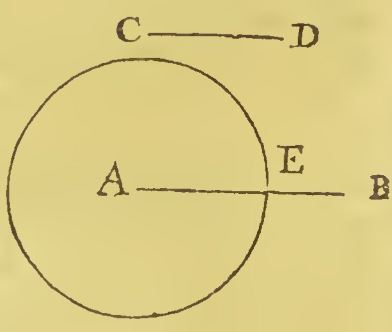
OF

GEOMETRY.

BOOK V.

PROBLEM I.

FROM the greater (AB) *of two unequal lines* (AB, CD) *to cut off, or take away a part* (AE) *equal to the leſſer* (CD).



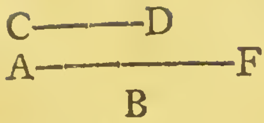
From A as a center, with a radius equal to CD, let the circumference of a circle be deſcribed [a], cutting [b] AB in E; *and the thing is done.*

[a] Poſt. 3.
[b] Ax. 2.

PROBLEM II.

At a given point (A) *to make a line* (AB) *equal to a given line* (CD).

Draw

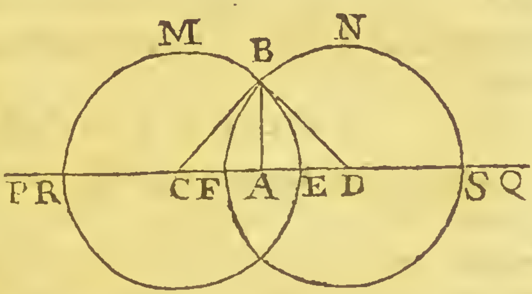


Draw the indefinite line AF [c]; from which take away AB = CD [d]; *and the thing is done.*

[c] Post. 1. and 2.
[d] 1. 5.

PROBLEM III.

At a given point (A) *in an infinite right-line* (PQ) *to erect a perpendicular.*



In the line propounded, take two equal distances AC, and AD [e]; and from the centers C [f] and D [g], with any equal radii greater than AC (or AD), let two circles EMR and FNS be described; which will cut each other: and, if from the point B of their intersection, you draw BA, *the thing is done.*

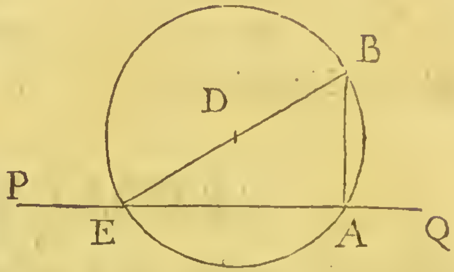
[e] 1. 5.
[f] Post. 3.
[g] Post. 1.

For let the points R, E, and F, S, be those wherein the infinite line PQ intersects the circumferences of the two circles EMR and FNS [h]: then AF being ⊐ FD [h] (or CR [i]) ⊐ AR [h]; and AS ⊏ DS [h] (or CE [i]) ⊏ AE [h], the point F falls within, and the point S without the circle EMR; and so the two circles cut each other [l]. If therefore, from the point of intersection, BC and BD be drawn; then the triangle CBD being isosceles [m], the angles BCD, BDC at the base thereof, will be equal [n]; and so, CA being = AD [k], and CB = BD [m], the angle CAB is also = DAB [p].

[h] Ax. 2.
[i] Hyp.
[l] Def. 11. 3.
[m] Constr. and Def. 33. of 1.
[n] *Obs. on* Ax. 10.
[k] Constr.
[p] Ax. 10.

Otherwise.

Otherwiſe.



From any point D above the line PQ, as a center, thro' the given point A, let the circumference of a circle be deſcribed[q], interſecting PQ in E[r]; draw the diameter EDB, and alſo BA[s]; then the angle EAB, being in the ſemicircle EAB[t], is a right-angle; *which was to be done.*

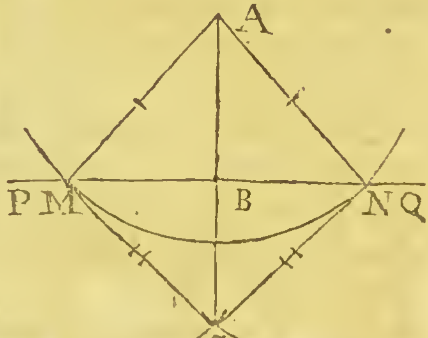
[q] Poſt. 3.
[r] Cor. to 4. 3.
[s] Poſt. 1.
[t] 13. 3.

COROLLARY.

From the former of theſe conſtructions it appears, that, if from any two points, with two equal radii, greater, each, than half the diſtance of thoſe points, two circles be deſcribed; thoſe circles will cut each other.

PROBLEM IV.

From a given point (A) *upon an infinite right-line* (PQ) *to let fall a perpendicular* (AB).



From the given point A, as a center, let an arch of a circle be deſcribed, ſo as to paſs below PQ and interſect it in M and N; from which points, with any equal radii, greater than half MN, let two other arches be alſo deſcribed, and from the point of

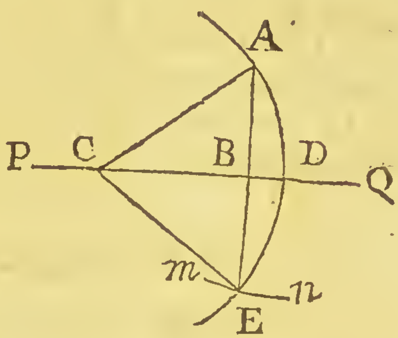
their

their interſection C, let the right-line CBA be drawn; which will be perpendicular to PQ.

For, let AM, AN, CM, and CN be drawn; then AM being = AN [u], and MC = NC [w], the angle AMB is = ANB [x], and CMB = CNB [x]; and conſequently AMC = ANC [y]: whence, (as AM = AN, and MC = NC) the triangles AMC, ANC are equal in all reſpects [z]; and ſo, the angle MAB being = NAB, the angle MBA is likewiſe = NBA [z].

[u] Def. 33. of 1.
[w] Hyp. and Def. 33. 1.
[x] *Obſ. on* Ax. 10.
[y] Ax. 4.
[z] Ax. 10.

The ſame otherwiſe.



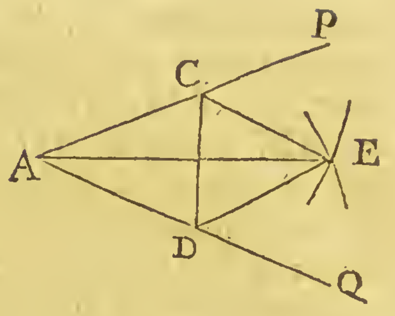
From any point C in the line PQ, as a center, let the circumference of a circle be deſcribed thro' the given point A [a], interſecting PQ in D [b]; and from the center D, with a radius equal to the diſtance of the points A and D, let another circle *m*E*n* be alſo deſcribed, cutting the former ADE in E; then draw ABE for the perpendicular required.

[a] Poſt. 3.
[b] Conſtr.

For, conceiving right-lines to be drawn from C and D, to A and E, the triangles ACE, ADE will be both of them iſoſceles [b]; and ſo the demonſtration is the ſame with that of the preceding method.

PROBLEM V.

To biſect, or divide into two equal parts, any given right lined angle (PAQ).



In the lines containing the given angle, take AC = AD[c]; and upon the centers C and D, with any equal radii, let two circles be deſcribed[d], ſo as to interſect each other; and from the point of interſection E draw EA, *and the thing is done.*

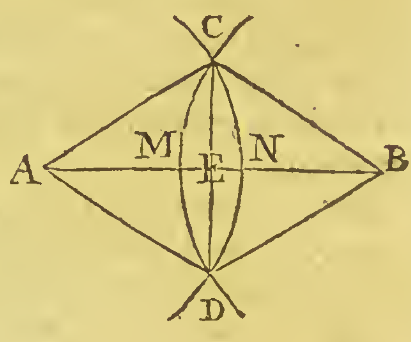
c 1. 5.
d Poſt. 3.

For, let CD, CE, and DE be drawn[e]; then, the triangles ACD and ECD being both iſoſceles[f], the angle ACD will be = ADC, and ECD = EDC[g]; and conſequently the whole angle ACE = the whole angle ADE[h]: whence (AC being = AD, and EC = ED), the angle CAE is alſo = the angle DAE[g].

e Poſt. 1.
f Conſtr. and Def. 33. of 1.
g Ax. 10.
h Ax. 4.

PROBLEM VI.

To biſect a given right-line (AB).



From the extremes A, B, of the given line, with equal radii, deſcribe two circles, ſo as to cut each other[i]; and between the two points of interſection draw CD, cutting AB in E; *and the thing is done.*

i Poſt. 3. and Cor. to 3. of 5.

For, if AC, AD, BC and BD be drawn, the triangles ACB, ADB being iſoſceles[k], thence is the angle CAB = CBA[l], and DAB = DBA[l]; and conſequently CAD = CBD[m]: whence the triangles ACD and BCD are equal in all reſpects[n]; and ſo the angle ACE being = BCE, AC = BC, and CE common, thence is AE alſo = BE[n].

k Conſtr. and Def. 33. of 1.
l *Obſ. on* Ax. 10.
m Ax. 4.
n Ax. 10.

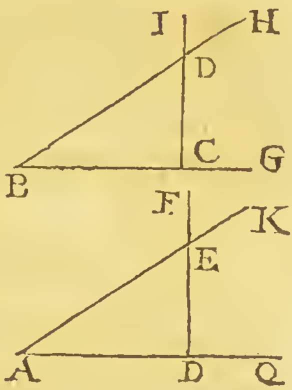
COROL-

COROLLARY.

Hence, it is manifeſt, that CD not only biſects AB, but is alſo perpendicular to it [n]. [n] Ax. 10. and Def. 8. of 1.

PROBLEM VII.

From a given point (A) *in a given right-line* (AQ) *to draw a line* (AK) *which ſhall make with the former an angle, equal to an angle given* (HBG).



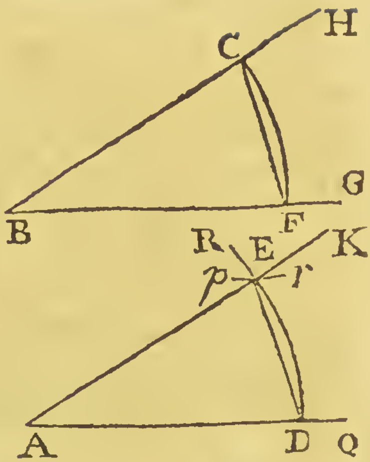
In BG and AQ take two equal diſtances BC, AD [o]; and at C and D erect the two perpendiculars CI and DF to BG and AQ [p]; and in DF take DE equal to the part CD of the former, intercepted by the lines containing the given angle HBG [q]; then thro' E draw AEK [r], *and the thing is done* [s].

[o] 2. 5.
[p] 3. 5.
[q] 1. 5.
[r] Poſt. 1.
[s] Ax. 10.

SCHOLIUM.

Having, in the ſeven preceding problems, effected and demonſtrated, *by means of the axioms only*, whatever was aſſumed in the fourth poſtulate, *as barely poſſible*; we are now authorized, by the moſt rigid laws of geometrical reaſoning, to make uſe of any theorem or concluſion, whatſoever, derived in the preceding books, in virtue of thoſe aſſumptions, by which the proceſs and reſult can be rendered the moſt obvious and eligible.—Accordingly, by having recourſe here to Theorem XIV. of the firſt book, we ſhall be able to arrive at a conſtruction of the laſt problem, better adapted to practice than that above laid down.

H 2 From

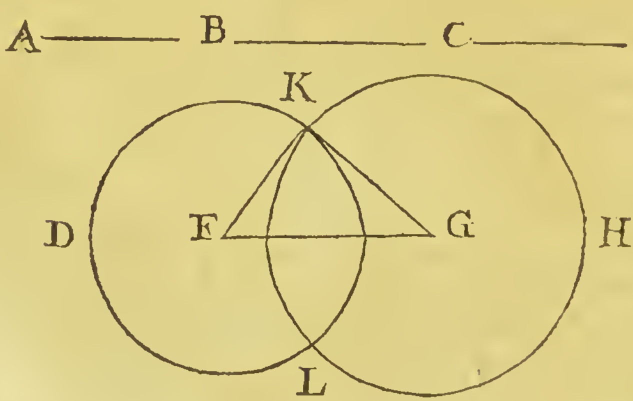


From the centers A and B, at any equal diſtances AD, BF, let two arcs of circles, FC, DR, be de-
[t] Poſt. 3. ſcribed [t], interſecting the given lines in D, F, and C; alſo from D, with a radius equal to the diſtance of the points F, C, let another circular arch *p*E*r* be deſcribed, cutting the former DR, in E; then draw AEK, *and the thing is done.*

For, conceiving right-lines to be drawn from F to C, and from D to E, the triangles BFC and ADE will be equilateral to each other, *by conſtruc-*
[u] 14. 1. *tion*, and therefore equiangular alſo. [u].

PROBLEM VIII.

To deſcribe a triangle, whoſe three ſides ſhall be equal to three given lines (A, B, C); *provided any two of them, taken together, be greater than the third.*



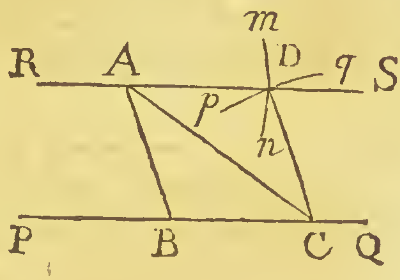
[w] 2. 5. Make FG = B [w], and from the centers F and G, with the intervals, or diſtances A and C, let two

two circles DKL, HKL be defcribed [x]; which will cut each other; and, if from the point of interfection K, the lines KF and KG be drawn, then will FKG be the triangle required. [x] Poft. 3.

For, the diftance FG of the two centers, is lefs than the fum of the radii A and C [y]; and greater than their difference (becaufe $B+A$ being $\sqsubset C$ [z]: thence is $B \sqsubset C - A$ [y]); therefore the two circles cut each other [a]: confequently $FK = A$, $FG = B$, and $GK = C$ [b]. [y] Ax. 6. 1. [z] Hyp. [a] 9. 3. [b] Def. 33. of 1.

PROBLEM IX.

Through a given point (A), *to draw a right-line* (RS) *parallel to a given right-line* (PQ).



At any point B in the given line, make BC equal to the diftance of the points A and B [c]: and from the centers A and C, with an interval equal to CB, let 2 circles be defcribed [d], then thro' their interfection D, let the line RS be drawn, *and the thing is done.* [c] 2. 5. [d] Poft. 3.

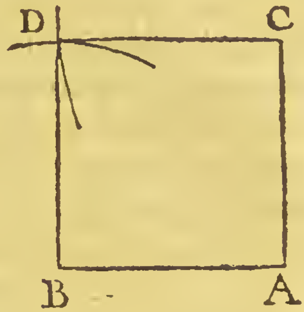
Let AB, and AC be drawn [e]. It is plain that the two circles will cut each other [f], becaufe the fum of their femi-diameters ($= AB + BC$ [g]) is greater than AC [h]: therefore, if ADS and CD be alfo drawn, then will $AB = BC = CD = DA$ [i], and therefore RS will be parallel to PQ [k]. [e] Poft. 1. [f] 9. 3. [g] Conftr. [h] 19. 1. [i] Ax. 1. [k] 25. 1.

The fame otherwife.

From A, to any point in PQ, draw AB [l]; make the angle $SAB = PBA$ [m]; and then AS will be parallel to PQ [n]. [l] Poft. 1. [m] 7. 5. [n] 8. 1.

PROBLEM X.

Upon a given line (AB) *to describe a square* (ABCD).



[o] 3. 5. and 1. 5. Make AC perpendicular, and equal to AB [o]; and from the centers B and C, let two circles, with the radius AB
[p] Post. 3. or AC, be described [p], inter-
[q] 9. 3. secting each other [q] in D; from which point draw DB and DC, *and the thing is done.*

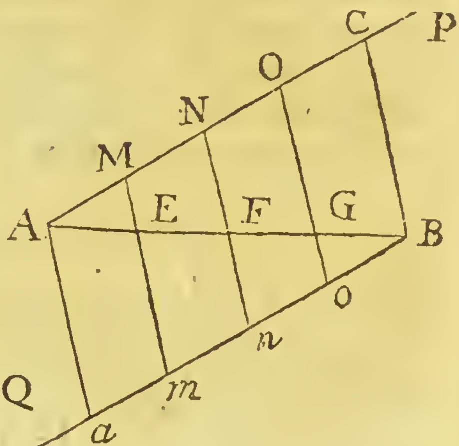
For, all the four sides being equal, by construc-
[r] 25. 1. tion, the figure is a parallelogram [r]; and therefore
[s] Constr. the angle A being a right-angle [s], the other three
[t] Cor. to 24. 1. will be all right-angles [t], and ACDB a square [u].
[u] Def. 26.

SCHOLIUM.

By the same method a rectangle may be described, the sides thereof being given.

PROBLEM XI.

To divide a given line (AB) *into any proposed number of equal parts.*

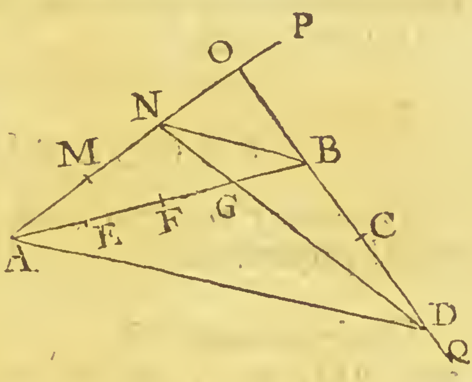


From the extremes of the given line AB, draw two indefinite lines AP, BQ parallel to each o-
[v] 9. 5. ther [v]; in each of which lines let there be taken as many equal distances AM, MN, NO, OC; Bo, *on*, *nm*, *ma*, (of any length

length at pleaſure) as you would have AB divided into[w]; then draw Mm, N*n*, O*o*, interſecting AB in E, F, G, *and the thing is done.* [w] 1. 5.

For, MN and *mn* being equal and parallel[x], FN will be parallel to EM[y]; and in the ſame manner will GO be parallel to FN: therefore, AM, MN, NO, *&c.* being all equal[x], AE, EF, FG, will likewiſe be equal[z]. [x] Conſtr. [y] 26. 1. [z] Cor. 1. to 27. 1.

The ſame otherwiſe.

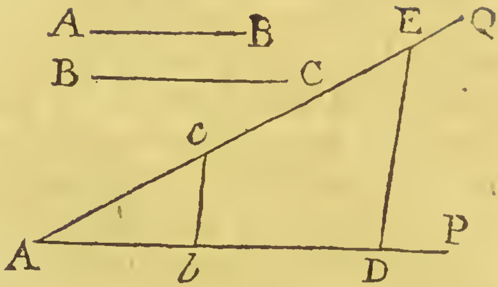


In any right-line AP, drawn from A, take as many equal diſtances (AM, MN, NO) wanting one, as you would have AB divided into; then, having drawn the indefinite line OBQ, in it take an equal number of parts or diſtances OB, BC, CD, each of the length of OB, and let DN be drawn, cutting AB in G; make GF, FE, each equal to BG, *and the thing is done.*

For, if AD and BN be drawn, they will be parallel[a], becauſe OA : ON : : OD : OB[b]); and ſo, the triangles BNG, ADG, being equiangular[c], it will be BG : AG : : BN : AD[d] : : ON : OA[d]. Therefore BG is the ſame part of AG, as ON is of OA. [a] Cor. to 12. 4. [b] Conſtr. [c] 3. and 7. 1. [d] 14. 4.

PROBLEM XII.

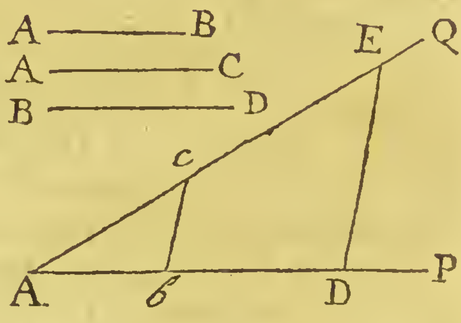
To two given lines (AB, BC) *to find a third proportional.*



From any point A, draw two indefinite lines AP, AQ, in which take A*b* = AB, A*c* = BC, and
[f] 1. 5. *b*D = BC [f]; draw *bc*, and parallel
[g] 9. 5. to *bc*, draw DE [g], cutting AQ in E; then *c*E will be the third proportional required: for, A*b* (AB) : A*c* (BC)
[h] 12. 4. : : *b*D (BC) : *c*E [h].

PROBLEM XIII.

To three given lines (AB, AC, BD) *to find a fourth proportional.*



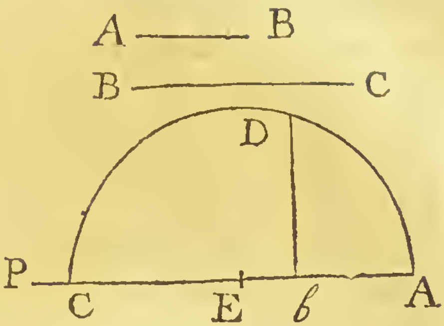
Having drawn AP and AQ, as in the preceding problem, take therein A*b* = AB, A*c* = AC, and
[i] 1. 5. *b*D = BD [i]; draw *bc*, and parallel to
[l] 9. 5. it, draw DE [l], interfecting AQ in E; then is *c*E *the fourth proportional required.*
[m] 12. 4. For, A*b* (AB) : A*c* (AC) : : *b*D (BD) : *c*E [m].

PROBLEM XIV.

Between two given lines (AB, BC) *to find a mean proportional.*

In

In the indefinite line AP, take $Ab = AB$, and $bC = BC$ [n]; bisect AC in E [o], and from the center E, at the distance of EA, or EC, let a semicircle ADC be described [p]; erect bD perpendicular to AC [q], cutting the circumference in D; then will bD be the mean-proportional required.



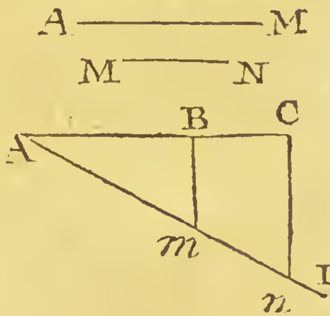
For, Ab (AB) : bD : : bD : bC (BC) [r].

[n] 1. 5.
[o] 6. 5.
[p] Post. 3.
[q] 3. 5.
[r] 19. 4. and it's Corol.

PROBLEM XV.

To divide a given line (AC) *into two parts* (AB, BC) *having the same proportion as two given lines* (AM, AN).

From A draw AD, making any angle with AB; in which take $Am = AM$, and $mn = MN$ [s]; draw nC, and mB parallel thereto [t], meeting AC in B. Then will AB : BC :: Am (AM): mn [u] (MN). *Which was to be done.*

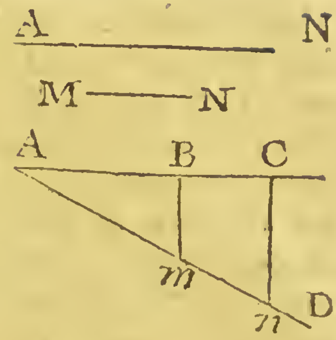


[s] 2. 5.
[t] 9. 5.
[u] 12. 4.

PROBLEM XVI.

To add a line (BC) *to a line given* (AB), *so that the whole compounded line* (AC) *shall be in proportion to the part added, as one given line* (AN) *is to another* (MN).

From

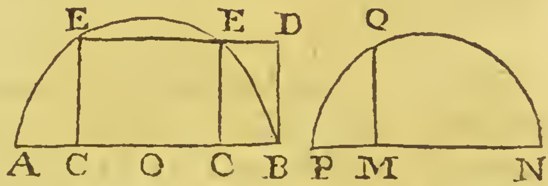


From A draw AD, making any angle with BA; in which [w] take $An = AN$ [w], and $nm = NM$; draw mB, and nC parallel thereto [x], meeting AB produced, in C; then will $AC : BC :: An\ (AN) : mn$ [y] (MN). *Which was to be done.*

[w] 2. 5.
[x] 9. 5.
[y] 12. 4.

PROBLEM XVII.

To divide a given line (AB) *into two such parts* (AC, BC) *that the rectangle contained under them, shall be equal to the rectangle under two given lines* (PM, MN); *provided that the given rectangle is not greater than the square of half the line* (AB) *to be divided.*



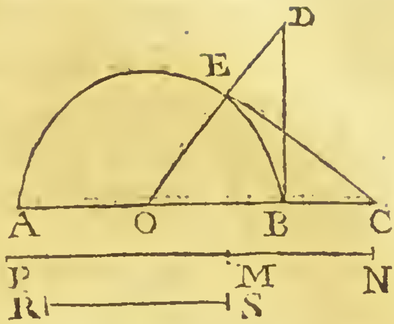
Between PM and MN take a mean-proportional MQ [z]; make BD perpendicular to AB, and equal to MQ; bisect AB in O [a], from which, as a center, let a semi-circle be described; and draw DE parallel to BA [b], which (because BD is less than the radius [c]) will meet the circle in some point E; from which, upon AB let fall the perpendicular EC: so shall $AC \times BC =$ [d] $EC^2 = DB^2\ (= QM^2) = PM \times MN$ [e]. *Which was to be done.*

[z] 14. 5.
[a] 6. 5.
[b] 9. 5.
[c] Hyp.
[d] Cor. to 19. 4.
[e] Hyp. and 10. 4.

PROBLEM XVIII.

To a given line (AB) *to add another line* (BC), *such, that the rectangle under the whole compounded line* (AC) *and the part added, shall be equal to a rectangle under two given lines* (PM, MN).

Between



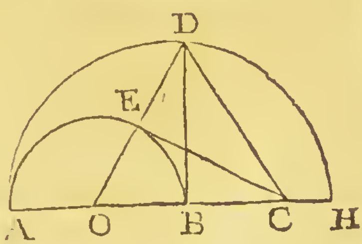
Between PM and MN take a mean proportional RS[f]; make BD perpendicular to AB[g], and equal to RS[h]; bisect AB in O[i], draw OD, and take $OC = OD$[h]: so shall $AC \times BC = PM \times MN$, *as was to be done.*

For, if thro' A and B, from the center O, the circumference of a circle be described, cutting DO in E, and E, C be joined; then, the triangles OCE, ODB (having $OE = OB$, $OC = OD$[k], and the angle EOB common) will be equal in all respects[l]; and so, EC being a[m] tangent to the circle in E, we have[n] $AC \times BC = CE^2 =$[o] $BD^2 = RS^2 = PM \times MN$[p].

[f] 14. 5. [g] 3. 5. [h] 2. 5. [i] 6. 5. [k] Constr. [l] Ax. 10. [m] 6. 3. [n] Cor. to 22. 3. [o] 1. 2. [p] 10. 4.

PROBLEM XIX.

To divide a given line (BH) *into two such parts, that the square of the one* (BC) *shall be equal to the rectangle under the other* (CH) *and a second given line* (AB).



Taking BA in the same strait line with BH, between them let a mean proportional BD be found[q]; bisect AB in O[r]; Draw OD, and make $OC = OD$; so shall $BC^2 = CH \times AB$, *as was to be done.*

For, by the demonstration of the precedent, $AC \times BC$ ($= CE^2 = BD^2$) $= AB \times BH$; from each of which taking away $AB \times BC$, there remains $BC^2 = AB \times CH$[s].

[q] 14. 5. [r] 6. 5. [s] 5. 2. and Ax. 5. 1.

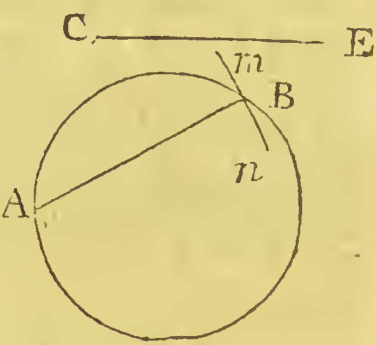
COROL-

COROLLARY.

If $AB = BH$, then will $BC^2 = BH \times CH$; in which case the line BH is said to be divided according to extreme and mean-proportion.

PROBLEM XX.

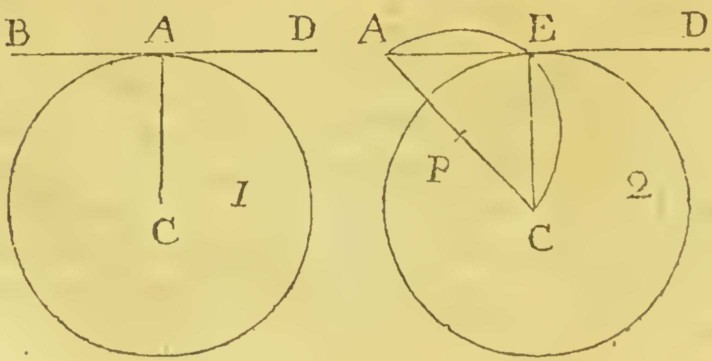
In a given circle, to apply, or inscribe a line (AB) *equal to a given line* (CE), *less than the diameter of the circle.*



From any point A in the circumference, with the radius CE, let a circu-
t Post. 3. lar arch *mn* be described [t], cutting the given circle in
u 9. 3. B [u]; then draw AB, *and*
w Def. 33. of 1. *the thing is done* [w].

PROBLEM XXI.

To draw a tangent to a given circle (C) *thro' a given point* (A).



CASE I. If the given point be in the circumference; then, to the center C, draw AC, and
y 3. 5. perpendicular to AC [y] draw BAD; which will
z 6. 3. touch the circle at A [z].

CASE II.

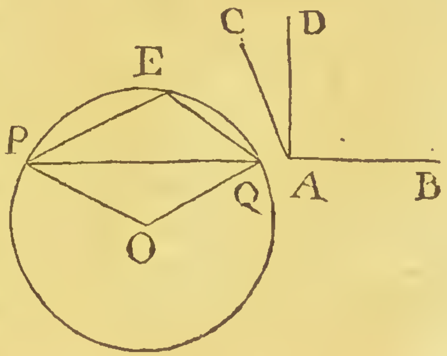
CASE II. If the point A be without the circumference; then draw AC, which biſect in P [b]; and from the center P, at the diſtance of AP, or CP, let a ſemi-circle AEC be deſcribed [c], cutting the given circle in E [d]; then draw AED, which will be the tangent required [e]; becauſe (CE being drawn) AEC is a right-angle [f].

[b] 6. 5. [c] Poſt. 3. [d] 9. 3. [e] 6. 3. [f] 13. 3.

PROBLEM XXII.

Upon a given line (PQ) *to deſcribe a ſegment of a circle* (PEQ) *to contain an angle* (E) *equal to a given angle* (BAC).

Make AD perpendicular to AB [g]; alſo make PQO, and QPO, each, equal to DAC [h] (the difference between the given angle and a right one); then upon the point of interſection O, as a center, at the diſtance of OP (or OQ), let a circle be deſcribed; *and the thing is done.*



[g] 3. 5. [h] 7. 5.

For the angle E = right-angle + QPO [i] = DAB + DAC [k] = BAC.

[i] 16. 3. [k] Conſtr.

SCHOLIUM.

In the ſame manner the problem may be conſtructed, when the given angle is acute; only the lines PO, QO muſt then be drawn on the other ſide of (PQ) as is manifeſt from the 16th theorem of the 3d book.

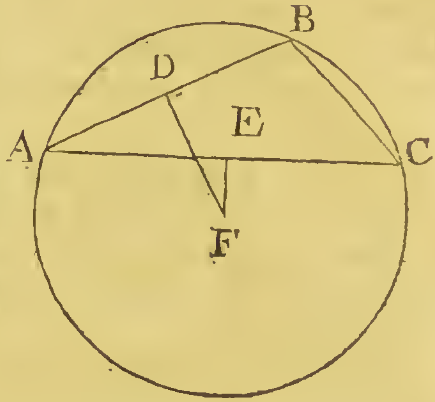
PRO-

PROBLEM XXIII.

About a given triangle (ABC) *to deſcribe a circle.*

Let any two ſides, AB and AC, be biſected by two perpendiculars DF and EF [k]; which will interſect each other in the center (F) of the required circle [l]; from whence the circle may be deſcribed.

k 6. 5.
l Proof of 19. 3.



SCHOLIUM.

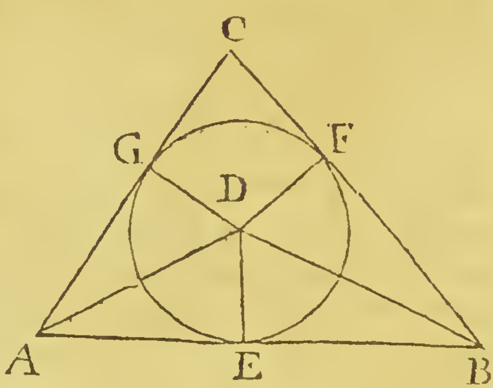
By the ſame method, the circumference of a circle may be deſcribed thro' any three given points, not ſituate in the ſame right-line: alſo from hence, the center of a circle may be found, by having a ſegment of the circle given.

PROBLEM XXIV.

To inſcribe a circle in a given triangle (ABC).

Biſect any two of the angles, A and B, by the lines AD and BD [m], meeting each other in D; make DE perpendicular to AB [o]; then, if from the center D, at the diſtance of DE, a circle be deſcribed, it will touch all the ſides of the triangle.

m 5. 5.
o 4. 5.



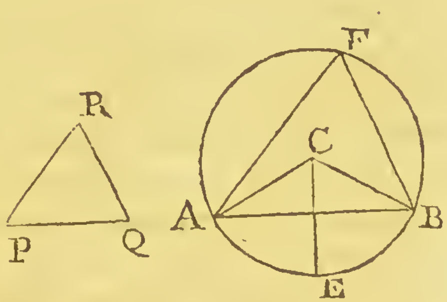
For,

For, let DG and DF be perpendicular to AC
and BC [p]; then the triangles ADE, ADG, having [p] 4. 5.
two angles equal, each to each (by construction)
and AD common, will not only be equiangular [q], [q] Cor. 1.
but also have DE = DG [r]. By the same argument to 10. 1.
DE = DF; therefore the circumference of the [r] 15. 1.
circle also passes through G and F [s]; but it touches [s] Def. 33.
the sides of the triangle in those points [t], because of 1.
G and F are right-angles [u]. [t] 6. 3.
[u] Constr.

PROBLEM XXV.

In a given circle (AFB) *to describe a triangle, equiangular to a given triangle* (PQR).

From the center C, draw the radii CA, CE, CB, making the angles ACE and BCE equal, each to the angle R [w]; join A, B, and make the angle ABF = Q [w], and from the point F, where BF cuts the circle, draw FA; so shall AFB be the triangle required.



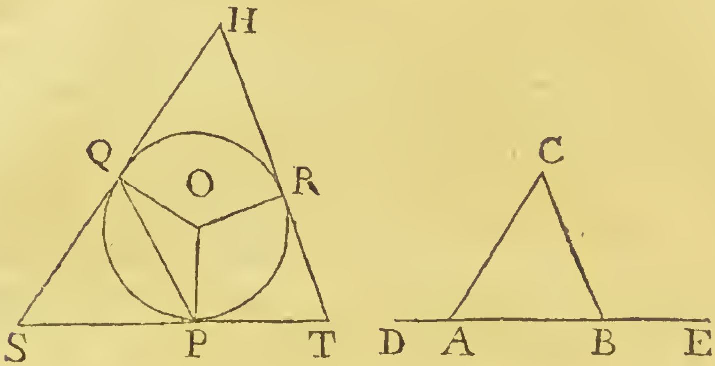
[w] 7. 5.

For, ABF = Q [x], F (= ACE [y]) = R [x]; and [x] Constr.
consequently BAF = P [z]. [y] 10. 3. and Constr.
[z] Cor. 1. to 10. 1.

PRO-

PROBLEM XXVI.

About a given circle (O), *to describe a triangle, equiangular to a given triangle* (ABC).



Produce out the side AB both ways; and draw the radii OP, OR, OQ, so as to make the angle
[a] 7. 5. POR = EBC, and POQ = DAC[a]; then draw three right-lines to touch the circle in the points
[b] 21. 5. P, Q and R[b], *and the thing is done.*

For, if PQ be drawn, the angles SQP and SPQ, will be less than the two right-angles SQO and
[c] Ax. 2. 1. and 6. 3. SPO[c]; and so PS and QS, not being parallels[d],
[d] Cor. 2. to 7. 1. and Cor. to 5. 1. they will meet each other[d]; therefore, as the like may be inferred with regard to PT and RT, &c. it is manifest that the three tangents form a triangle STH. Now, POR + T being = two right-
[e] Cor. 2. to 11. 1. angles[e] = ABC + EBC[f], and POR = EBC[g]; thence will T = ABC: and, by the same argu-
[f] 1. 1. ment, S = BAC; whence also H = C.
[g] Constr.

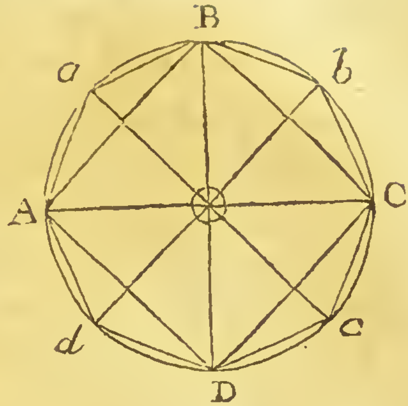
PROBLEM XXVII.

In a circle given (ABCD) *to inscribe a square.*

Draw two diameters AC and BD perpendicular
[h] 3. 5. to each other[h]; then draw AB, BC, CD and DA; so shall ABCD be a square inscribed in the circle.

For,

For, the angles AOB, BOC, DOC and DOA (as well as the fides OA, OB, OC, OD, containing them) being equal [i], the oppofite fides AB, BC, CD, DA will likewife be equal [k]: and the angles ABC, BCD, CDA, DAB, are all of them right-angles [l], and therefore are equal.



[i] Ax. 7. 1. and Def. 33. 1.

[k] Ax. 10. of 1.

[l] 13. 3.

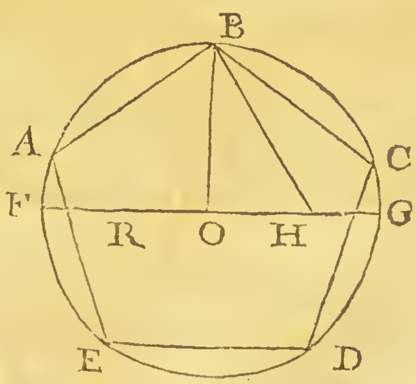
SCHOLIUM.

If two other diameters *ac*, *bd* be drawn (*by prob.* 5.) to bifect the angles AOB, BOC, a regular octagon A*a*B*b*C*c*D*d* may be infcribed in the circle. And if all the angles at the center O, be again bifected, a regular polygon of fixteen fides, may in like manner be determined; and fo on, at pleafure.

PROBLEM XXVIII.

In a circle given (ABGE) *to infcribe a regular pentagon.*

At the center O, upon the diameter FG, erect the perpendicular OB [m], meeting the circumference in B; divide OG in H (*by prob.* 19.) fo that $OH^2 = GH \times OG$; that is, take $OR = \frac{1}{2}OF$, and RH = dift. RB: Then draw BH; which will be equal to the fide of the pentagon [n]; from whence the figure itfelf may be defcribed.



[m] 3. 5.

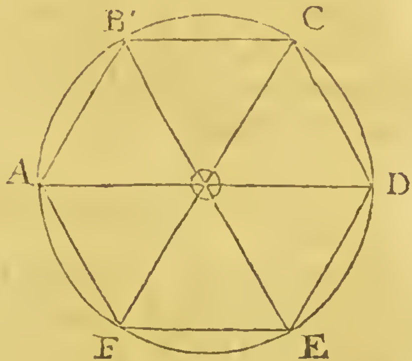
[n] 28. 3. and 8. 2.

SCHOLIUM.

[o] 28. 3. Hence a regular decagon may be inſcribed in the circle; the ſide thereof being = OH[o].

PROBLEM XXIX.

In a circle given, to inſcribe a regular hexagon (ABCDEF.)



From the extremes of any diameter AD, apply AB, AF, DC, and DE equal, each, to
[p] 20. 5. the radius AO[p]; then join B, C, and E, F; *and the thing is done.*

For, if the radii OB, OC, OE, OF be drawn; the triangles AOB and
[q] Conſtr. DOC, being equilateral[q], will alſo have the an-
[r] 14. 1. gle OAB = DOC[r]; whence AB is parallel (as
[s] Cor. to 8. 1. well as equal) to OC[s]; and conſequently BC and AO are likewiſe equal, and parallel[t]: There-
[t] 26. 1. fore, ſeeing the triangles AOB, BOC, COD, &c. are equilateral, and alike in all reſpects; not only the ſides, but alſo the angles ABC, BCD, &c. of
[u] Ax. 4. 1. the hexagon, will be equal among themſelves[u].

COROLLARY.

Hence it appears, that the ſide of a regular hexagon, inſcribed in a circle, is equal to the ſemi-diameter, or radius.

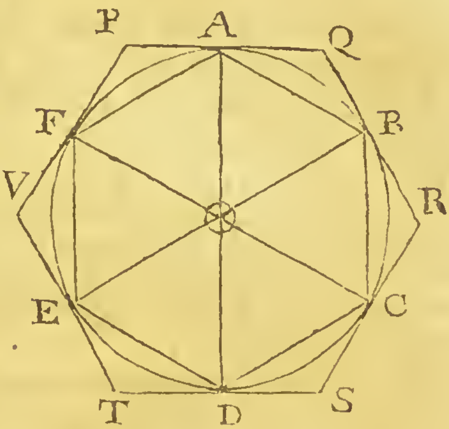
SCHOLIUM.

Beſides the figures conſtructed in the preceding problems, and thoſe ariſing from thence by continual biſections, or taking the differences, no other regular polygon can be deſcribed, from any known method, *purely geometrical*, by means of right-lines and circles only.

PRO-

PROBLEM XXX.

About a given circle to deſcribe a regular polygon, of the ſame number of ſides with a regular polygon (ABCDEF) *inſcribed in the circle.*

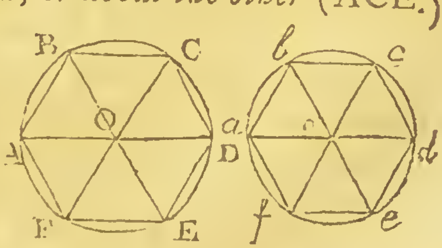


From the center O, to the angles of the inſcribed polygon, draw OA, OB, OC, &c. and perpendicular thereto draw PAQ, QBR, RCS[t], &c. interſecting[u] in P, Q, R, S, T, V; ſo ſhall PQRSTV be the polygon that was to be deſcribed. [t] 3. 5. [u] Proof of 26. 5.

For, by taking away the equal[w] angles OAF, OAB, OBA, OBC, &c. from the equal (right) angles OAP, OAQ, OBQ, OBR, &c. the remainders FAP, BAQ, ABQ, CBR, &c will alſo appear to be equal[x]: therefore the triangles FAP, ABQ, BRC, &c (having alſo FA = [w] AB = BC, &c.) are equal in all reſpects[y]; and ſo the angles P, Q, R, &c. as well as the ſides PQ, QR, RS, &c. are equal among themſelves[z]. [w] Hyp. [x] Ax. 5. 1. [y] 15. 1. [z] Ax. 4. 1

PROBLEM XXXI.

Any two circles (ACE, ace) *being given, to deſcribe a polygon in, or about the one* (ace) *that ſhall be ſimilar to any polygon deſcribed in, or about the other* (ACE.)



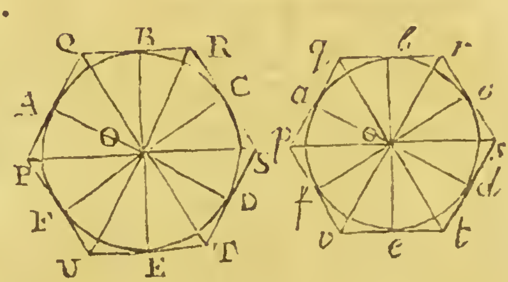
Firſt, having drawn the radii OA, OB, &c. to the angles of the given inſc. polyg. ABCDEF; make, at the center *o*, the angle *aob* = AOB, *boc* = BOC[a], &c. Then, the chords *ab*, *bc*, *cd*, &c. being drawn, I ſay, the polygon *abcdef* will be ſimi- [a] 7. 5.

 lar

lar to the given one ABCDEF. For, the triangles AOB, *aob*; BOC, *boc*; &c. being equiangular[b], the angles ABC, *abc* muſt alſo be equal[c]; and[d] AB : *ab* (: : OB : *ob*) : : BC : *bc*. In the ſame manner, the other correſponding angles are equal, and the ſides containing them proportional: Therefore the two polygons are ſimilar[e].

[b] Cor. 1. to 10. 1. and 12. 1.
[c] Ax. 4. 1.
[d] 14. 4.
[e] Def. 14. 4.

Again, having drawn the radii OA, OB, OC, &c. to the points of contact of the given circumſcribing polygon PQRSTU; draw likewiſe the radii *oa*, *ob*, *oc*, &c. making the angle *aob* = AOB, *boc* = BOC[f], &c. perpendicular to which[g] draw *pq*, *qr*, *rs*, &c. ſo ſhall the polygon *pqrstv* be ſimilar to the given one PQRSTU.



[f] 7. 5.
[g] 3. 5.

For, the angles OAQ, OBQ, *oaq*, *obq*, being all equal[h], and AOB alſo = *aob*[i]; the remaining angles AQB, *aqb* of the two quadrilaterals AOBQ, *aobq* muſt be equal[k]; as muſt likewiſe their halves OQB, *oqb* (for the right-angled triangles OAQ, OBQ, having OA = OB, and OQ common, have alſo OQA = OQB[l]). In the ſame manner is ORB = *orb*, &c. whence, the triangles POQ, *poq*; QOR, *qor*, &c. being equiangular, it follows[m] that PQ : *pq* (: : OQ : *oq*) : : QR : *qr* : And ſo of the reſt. Therefore PQRSTU and *pqrstv* are ſimilar[n].

[h] Ax. 7. 1.
[i] Conſtr.
[k] 11. 1. and Ax. 5.
[l] 16. 1.
[m] 14. 4.
[n] Def. 14.

COROLLARY.

It appears from hence, that the ſimilar inſcribed polygons, as well as the circumſcribing ones, are in proportion, as the ſquares of the radii of their reſpective circles. For, in the former caſe, $AO^2 : ao^2 ::$ [o] $AB^2 : ab^2 ::$ [p] ABCDEF : *abcdef*; and, in the latter, [o] $AO^2 : ao^2 :: PO^2 : po^2 :: PQ^2 : pq^2 ::$ [p] PQRSTU : *pqrstv*.

[o] 14. 4. and Cor. to 11. 4.
[p] 26. 4.

The End of the FIFTH BOOK.

ELEMENTS

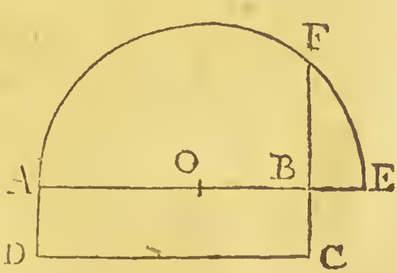
OF

GEOMETRY.

BOOK VI.

PROBLEM I.

To make a ſquare equal to a given rectangle (ABCD).



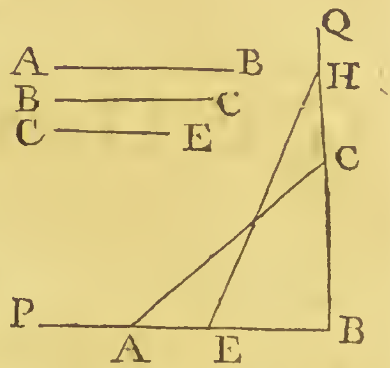
IN one ſide AB of the rectangle, produced, take BE = the other ſide BC; biſect AE in O [a]; and from the center O, at the diſtance of OA, or OE, let a ſemi-circle AFE be deſcribed; and let CB be produced to meet the circumference thereof in F; then a ſquare deſcribed on BF (by 10. 5.) will be equal to the given rectangle ABCD [b].

[a] 6. 5.

[b] Cor. to 19. 4.

PROBLEM II.

To make a ſquare equal to the ſum of two given ſquares.



Let AB and BC be the ſides of the two given ſquares.

[c] 3. 5. Draw two indefinite lines BP, BQ, at right-angles to each other[c]; in which take BA = BA, BC = BC, and join A, C; then a ſquare deſcribed on AC (*by* 10. 5.) will be equal to the ſum of the two ſquares deſcribed

[d] 8. 2. upon AB and BC[d].

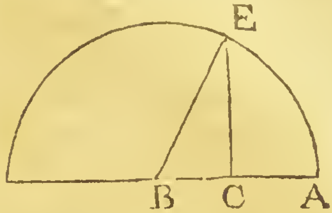
SCHOLIUM.

In the ſame manner a ſquare may be made equal to the ſum of three, or more, given ſquares: for if AB, BC, CE be taken as the ſides of the given ſquares, then, by making BH = AC, BE = CE, and drawing EH, it is evident that a ſquare upon EH will be equal to the ſum of the three ſquares upon AB, BC, and CE; or that, $EH^2 = BH^2$ $(AC^2) + BE^2 = AB^2 + BC^2 + CE^2$.

PROBLEM III.

To make a ſquare equal to the difference of two given ſquares.

Let



Let AB and BC (taken in the fame ftrait line) be equal to the fides of the two given fquares.

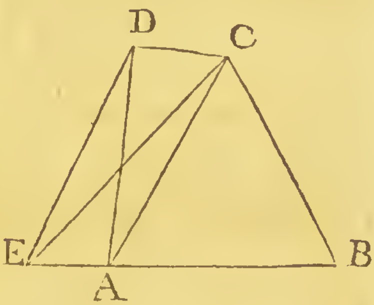
Upon the center B, with the radius BA, let a circle be defcribed, and make CE perpendicular to BC [e], meeting the circumference thereof in E: fo fhall a fquare defcribed on CE (*by* 10. 5.) be equal to BE^2 (BA^2) — [f] BC^2.

[e] 3. 5.

[f] Cor. to 8. 2.

PROBLEM IV.

To make a triangle equal to a given quadrilateral (ABCD.)



Draw the diagonal AC, alfo draw DE parallel to AC [g], meeting BA produced in E, and join CE; then will the triangle BCE = the given quadrilateral ABCD.

For, the triangles ACE, ACD, being upon the fame bafe AC, and between the fame parallels AC and ED, are equal [h]; therefore, if ABC be added to each, then alfo will BCE = ABCD [i].

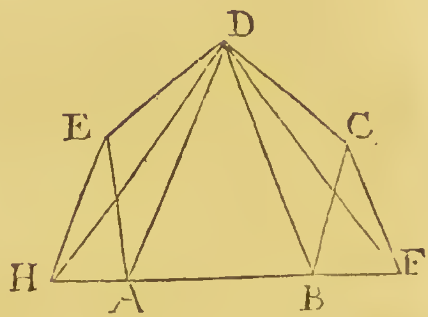
[g] 9. 5.

[h] Cor. 1. to 2. 2.

[i] Ax. 4. 1.

PROBLEM V.

To make a triangle equal to a given pentagon (ABCDE).



Draw DA and DB, and alſo EH and CF parallel to them [k], meeting AB produced in H and F; then draw DH and DF; ſo ſhall the triangle DHF = the pentagon ABCDE.

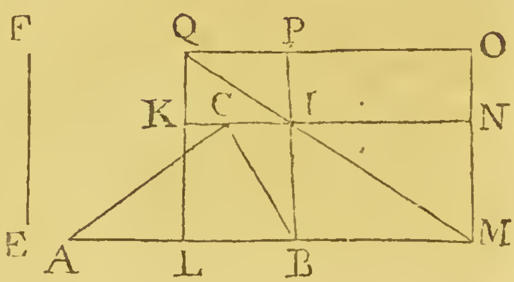
k 9. 5.

For the triangle DHA is = DEA [l], and DFB = DCB [l]; therefore DHF (= DHA + DAB + DFB = DEA + DAB + DCB) = ABCDE [m].

l Cor. 1. to 2. 2.
m Ax. 4. 1.

PROBLEM VI.

Upon a given line (EF), *to make a rectangle equal to a given triangle* (ABC).



Thro' C, the vertex of the triangle, draw KN parallel to the baſe AB [n]; and biſect AB with the perpendicular LQ [o], meeting KN in K; alſo draw BP perpendicular to AB [p], interſecting KN in I; then in AB, produced, take BM = EF, and draw MIQ cutting LQ in Q; draw QO and MO, parallel to AM and LQ [n], meeting each other in O: then will INOP be the rectangle required.

n 9. 5.
o 6. 5.
p 3. 5.

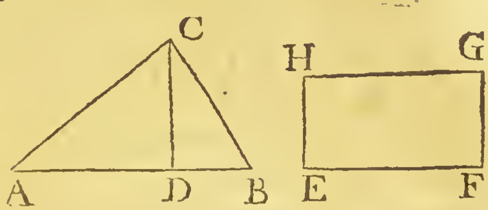
For,

For, it is evident, that LI, IO and LO are all rectangles[q]: therefore IN = BM[r] = EF[s], and IO = LI[t] = ABC[u].

[q] Cor. to 24. 1.
[r] 24. 1.
[s] Constr.
[t] 3. 2.
[u] Cor. 2. to 2. 2.

The same otherwise.

From the vertex C, upon the base AB, let fall the perpendicular CD[w]; make EH perpendicular to EF[x], and equal to a fourth proportional to 2EF, AB, and CD[y]: then the rectangle EG contained under EF and EH will be equal to the triangle ABC.



[w] 4. 5.
[x] 3. 5.
[y] 13. 5.

For, since, by construction, 2EF : AB :: CD : EH, therefore is 2EF × EH = AB × CD[z], and consequently EF×EH = ½ AB×CD = ABC[a].

[z] 10. 4.
[a] Cor. 2. to 2. 2.

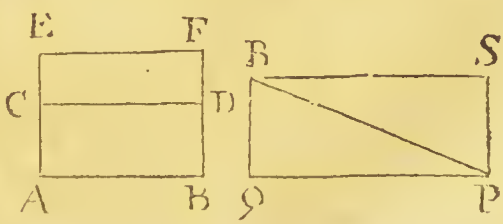
SCHOLIUM.

By either of the two preceding methods, a parallelogram having a given angle may be described upon a given line, equal to a given triangle; if, instead of MBP, MLQ, or BDC, FEH being right angles, you make them all equal to the angle given: the rest of the construction being the same.

PROBLEM VII.

Upon a given line (AB) *to describe a rectangle equal to a given right-lined figure* (PQRS).

Let the given figure be divided into triang. PQR, PRS: and upon the given line AB (by the precedent) let a rectangle ABDC, equal to the triangle PQR, be



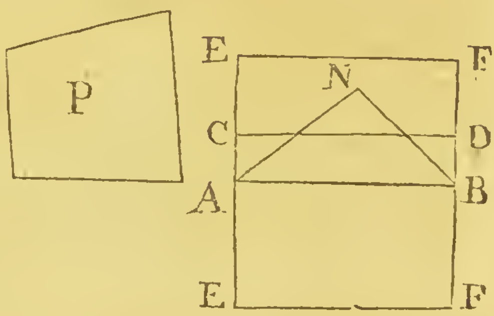
be deſcribed; alſo upon CD make the rectangle CDFE equal to the triangle PRS: ſo ſhall ABEF, which is a rectangle (becauſe both ACE, and
[b] 2. 1. BDE are continued right-lines [b]) be = PQR +
[c] Ax. 3. and 4. 1. PRS [c] = PQRS; *which was to be done.*

SCHOLIUM.

When the figure given has not more than five ſides, the conſtruction will be more eaſy, by firſt finding a triangle equal to it (*by Prob.* 4. *or* 5.) and then making a rectangle equal to that triangle. But if the figure be a rectangle, the eaſieſt way of all, will be to take a fourth-proportional BF to the given line AB and the two ſides PQ and PS of the given rectangle (*by* 13. 5.); which fourth-proportional will be the altitude of the rectangle required. For, ſince AB : PQ : : PS : BF (*by Conſtr.*) therefore (*by* 10. 4.) $AB \times BF = PQ \times PS$.

PROBLEM VIII.

To deſcribe a rectangle equal to the ſum, or difference of two given right-lined figures.



Let the two given figures be ABN and P.

By the precedent, let two rectangles AD and AF, reſpectively equal to ABN and P, be deſcribed on the ſame, or different ſides of AB, according as the difference, or ſum, of the two figures is required: then will the rectangle CF be equal to
[c] Ax. 4. or 5. 1. that ſum or difference [c].

COROL-

COROLLARY.

Hence two lines having the ſame ratio with two given right-lined figures, are determined: for AC : AE : : AD (ABN) : AF[f] (P). [f] 7. 4.

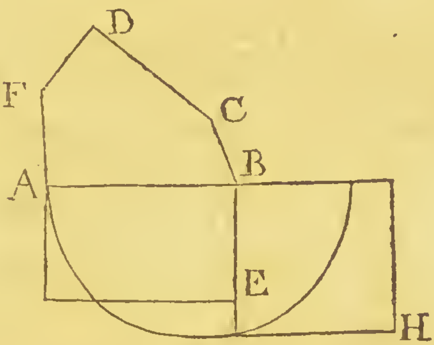
PROBLEM IX.

To make a ſquare equal to any right-lined figure given (ABCDF).

Upon AB deſcribe a rectangle AE equal to ABC DF[g]; then make a ſquare BH equal to that rectangle[h], *and the thing is done.*

[g] 7. 6.

[h] 1. 6.



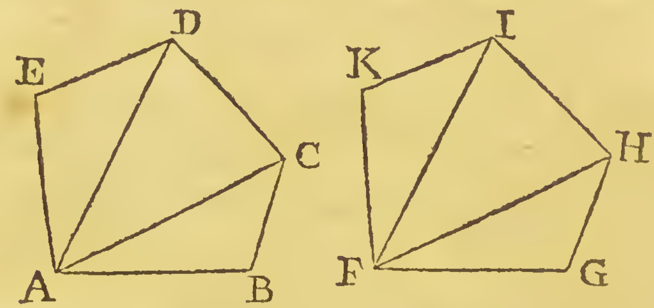
SCHOLIUM

After the ſame manner (*from prob.* 8.) a ſquare may be deſcribed equal to the ſum, or difference, of any two given right-lined figures.

PRO-

PROBLEM X.

To describe a figure (FGHIK) *equal, and similar to a given right-lined figure* (ABCDE).



[i] 2. 5. [k] 7. 5. Draw AC and AD, and also FG equal to AB [i]; make the angle GFH = BAC [k], HFI = CAD, and IFK = DAE [k]; likewise make FH = AC, FI = AD, and FK = AE [i]; then draw GH, HI, and IK, *and the thing is done.*

[l] Ax. 10. 1. For, since the triangle FGH = ABC [l] and FHI = ACD [l], *&c*; therefore is the whole polygon FGHIK, also, equal to the whole polygon [m] Ax. 4. 1. ABCDE [m].

Moreover, these equal triangles being also equiangular [l], it is manifest, that G = B, GHI = BCD [m], HIK = CDE, and so on; therefore, FG being also = AB, GH = BC, HI = CD [l], *&c.* the two polygons ABCDE, FGHIK are similar to each [n] Def. 14. 4. other [n].

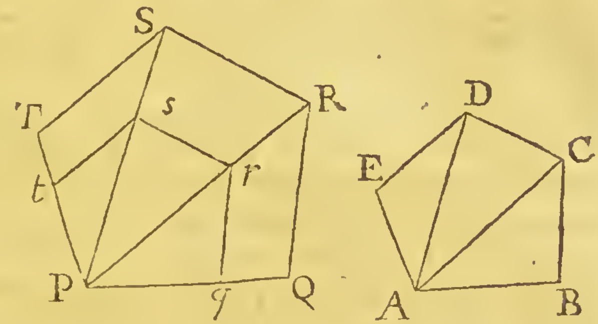
SCHOLIUM.

The figure FGHIK may be otherwise constructed, by making the triangles FGH, FHI, *&c.* respectively equilateral to ABC, ACD, *&c.* as is evident from 14. 1. and Ax. 4.

PRO-

PROBLEM XI.

Upon a given line (AB) *to describe a figure* (ABCDE) *similar to a given right-lined figure* (PQRST).



Draw PR and PS, and in PQ (produced if needful) take P*q* = AB; draw *qr* parallel to QR [o], meeting PR in *r*; also draw *rs* and *st*, parallel to RS and ST [o], intersecting PS and PT in *s* and *t*; then upon AB, *by the precedent*, describe a polygon equal and similar to P*qrst*; *and the thing is done.* [o] 9. 5.

For, since any angle BCD (*qrs*) of the polygon ABCDE, is equal to its correspondent QRS [p]; and also CB (*qr*) : CD (*rs*) : : RQ : RS [q], therefore the two polygons ABCDE, PQRST are like to each other [r].

[p] Cor. 1. to 7. 1.
[q] 21. 4.
[r] Def. 14. of 4.

SCHOLIUM.

This last problem may be otherwise constructed by making the triangles ABC, ACD, ADE equiangular to the triangles PQR, PRS, PST, respectively.

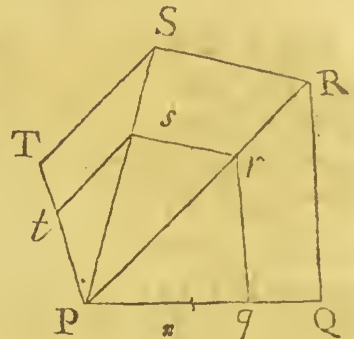
For, then the angle BCD being = QRS, CDE = RST [s], &c.; and also BC : QR (: : AC : PR) : : CD : RS [t], &c. the two polygons must therefore be similar to each other [u].

[s] Ax. 4. 1.
[t] 14. 4.
[u] Def. 14. of 4.

PRO-

PROBLEM XII.

To deſcribe a figure, ſimilar to a right lined figure given (PQRST), *which ſhall be to it in a given ratio of one right-line to another.*

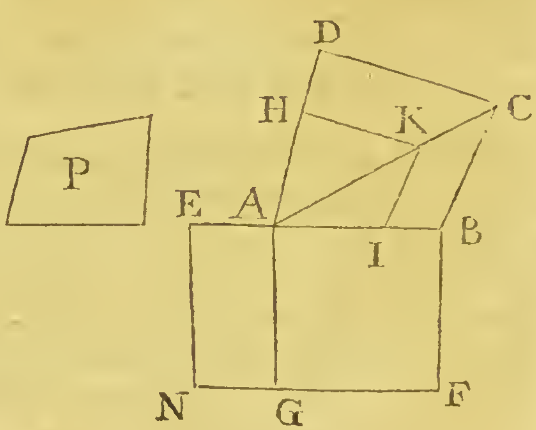


In PQ (produced if neceſſary) take P*n* to PQ in the given ratio of the figure to be deſcribed to the figure
w 12. 5. given [w]; and, in the ſame line PQ, take P*q* equal to a mean-proportional between
x 14. 5. P*n* and PQ [x]; upon which (*by the precedent*) let P*qrst*, ſimilar to PQRST, be deſcribed, *and the thing is done.*

For, ſince P*n* : P*q* : : P*q* : PQ (*by Conſtr.*);
y 27. 4. therefore P*n* : PQ : : P*qrst* : PQRST [y].

PROBLEM XIII.

To deſcribe a figure that ſhall be equal to one right-lined figure given (P), *and ſimilar to another* (ABCD).



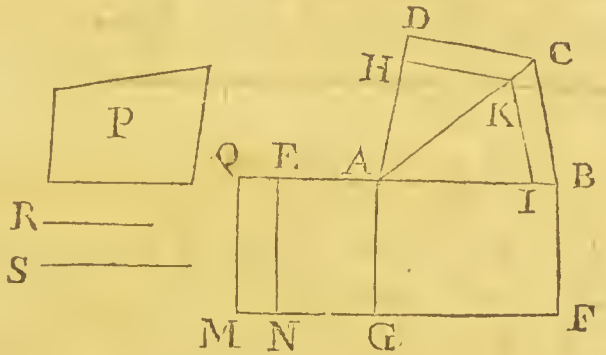
Upon AB make the rectangle ABFG
z 7. 6. = ABCD [z], and upon AG make the rectangle AGNE = P [z]; in AB take AI equal to a mean proportional between AB and
a 14. 5. AE [a]; and upon AI let AIKH be deſcribed ſimilar,

lar, and alike ſituate, to ABCD[b], *and the thing is done.* [b] 11. 6. [c] 7. 4.

For ABCD (AF) : P (AN) : : AB : AE[c] : : ABCD : AIKH[d]; and therefore AIKH = P[e]. [d] 27. 4. [e] Ax. 4. 4.

PROBLEM XIV.

To deſcribe a figure ſimilar to a given right-lined figure (ABCD), *which ſhall be to another given right-lined figure* (P) *in a given ratio of one right-line* (S) *to another* (R).



Make the rectangle ABFG = ABCD[g], and the rectangle AGNE = P[g]; alſo in AE, produced, take AQ = a fourth-proportional to R, S and AE[h]; then, *by the precedent*, make AIKH ſimilar to ABCD, and equal to the rectangle AG × AQ: then will AIKH (AG × AQ) : P (AG × AE[i]) : : AQ : AE[k] : : S : R[i]; *which was to be done.* [g] 7. 6. [h] 13. 5. [i] Conſtr. [k] 7. 4.

The End of the SIXTH BOOK.

ELEMENTS

OF

GEOMETRY.

BOOK VII.

DEFINITIONS.

1. A Right-line is ſaid to be perpendicular to a plane, when it is perpendicular to all right-lines, that can be drawn in that plane, from the point on which it inſiſts.

2. One plane is ſaid to be perpendicular to another, when all right-lines drawn in the one, perpendicular to the common ſection, are perpendicular to the other.

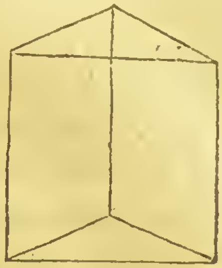
3. Parallel planes are thoſe, which are every where equally diſtant, the one from the other.

4. A Solid is that, which has length, breadth, and thickneſs.

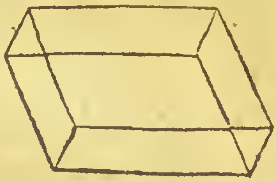
5. Similar ſolids are ſuch, as are bounded by an equal number of ſimilar planes.

6. A Priſm

6. A Prifm is a folid, whereof the planes of the fides are parallelograms, and whereof the two ends, or oppofite bafes, are plane, rectilinear figures, parallel to each other.

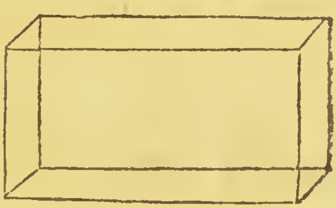


7. A parallelepipedon is a folid bounded by fix parallelograms, whereof the oppofite ones are parallel, equal, and like to each other.

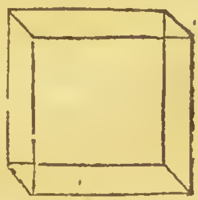


8. An upright prifm, or parallelepipedon, is that, whereof the planes of the fides are perpendicular to the plane of the bafe.

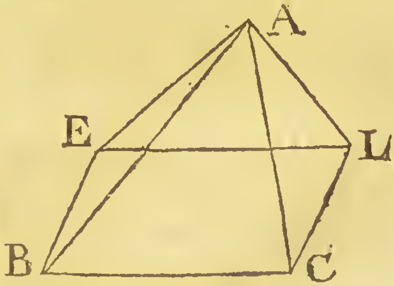
9. A rectangular parallelepipedon is that, whofe bounding planes are all rectangles, and which ftand at right-angles one to another.

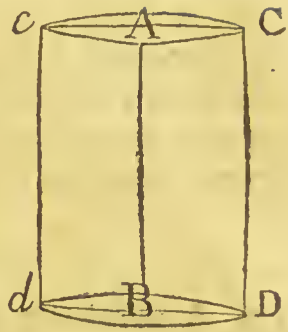


10. When all the bounding planes are fquares, the parallelepipedon is called a cube.

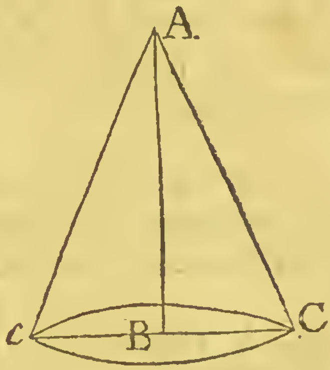


11. A Pyramid is a folid, whofe bafe is any right lined plane figure, and whofe fides are triangles, having all their vertices united in a point, above the bafe, called the vertex of the pyramid. Thus ABCLE reprefents a pyramid, whofe vertex is A, and bafe BCLE.





12. A Cylinder (DC*cd*) is a ſolid generated by the rotation of a rectangle ACDB about one of its ſides AB, ſuppoſed at reſt; which quieſcent ſide AB is called the axis of the cylinder.



13. A cone (AC*c*) is a ſolid generated by the rotation of a right-angled triangle ABC about its perpendicular AB, called the axis of the cone.

14. A Sphere is a ſolid generated by the rotation of a ſemi-circle about its diameter.

15. The Fruſtum of a pyramid, or cone, is that part which remains, when any part next the vertex, cut off by a plane parallel to the baſe, is taken away.

16. The altitude of a pyramid, or priſm, is the perpendicular diſtance of the vertex, or upper plane thereof, from the plane of the baſe.

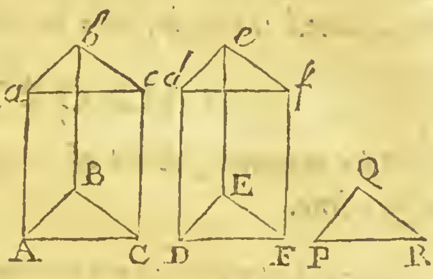
17. Every rectangular parallelepipedon is ſaid to be contained under the three right-lines that are the length, breadth, and altitude thereof.

18. A Plane is ſaid to be extended (or to paſs) by a right-line, when every part of the latter is placed in, or touched by, the former.

An

An AXIOM

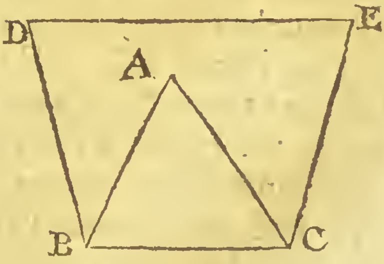
Upright priſms (A*abc*CBA, D*def*FED) of the ſame altitude, ſtanding upon baſes (ABC, DEF) equal and like to each other, are themſelves equal.



To ſee the evidence of this AXIOM *in the ſtrongeſt light, conceive a right lined plane figure* PQR *to be formed, equal and like in all reſpects to the baſes* ABC, DEF *of the two priſms; upon which, conceive the priſms to be placed, one after another, ſo that their baſes may coincide therewith. Then, becauſe the planes of the ſides ſtand, in both caſes, perpendicular to the plane of the baſe, upon the ſame lines* PQ, QR, PR, *and are carried up to the ſame height, it is manifeſt, that the bounds of the two ſolids, when thus placed, have the very ſame poſition; and, conſequently, that the ſolids themſelves, occupying (ſucceſſively) the ſame identical ſpace, are equal the one to the other.*

A POSTULATE.

That by any two right-lines (AB, AC) meeting in a point, a plane may be extended.



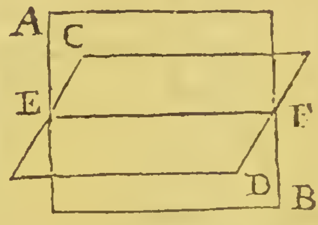
In order the better to comprehend the ſenſe and deſign of this Poſtulate, *let a plane* BDEC, *extended by the right-line joining the points* B *and* C, *be conceived to be revolved about upon that line, till it meets with, or takes in, the point* A; *then the plane including, in that poſition, all the*

the three points B, C, *and* A, *it also includes, or is extended by, the right lines* AB, AC, BC, *joining those points; which are in the same plane with their extremes* (*by def* 6. 1.)

Hence it appears, that, by any three points, a plane may be extended; and that all the three sides of any right-lined triangle, are in the same plane.

THEOREM I.

The common section of two planes (AB, CD) *is a right line.*

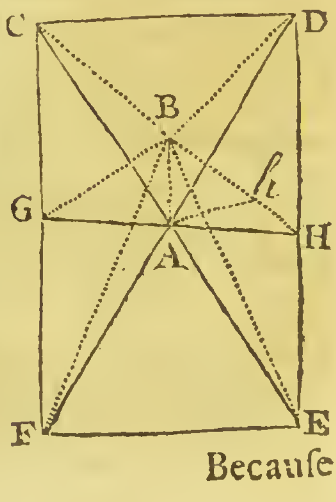


For, between the two extreme points E, F of the common section, let a right-line EF be drawn[a]; then, that line being in the plane AB[b], and also in the plane CD[b], it must, of consequence, be the common section of them both.

[a] Post. 1.
[b] Def.6.1. and Ax. 8. 1.

THEOREM II.

If a right-line (AB) *be perpendicular to two other right-lines* (CE, DF) *cutting each other, at the common section* (A), *it will be perpendicular to the plane* (CDEF) *passing by those two lines.*



Take AC, AD, AE, AF all equal to one another; and, having joined CD, DE, EF, CF, let there be drawn thro' A, in the plane CDEF, any right-line GH, meeting CF and DE in G and H; and let BC, BG, BF, BD, BH, and BE be also joined.

Because

Becaufe AC [b] = AE = AD = AF, and CAF = DAE [c], therefore is CF = DE [d], and the angle FCA (or GCA) = DEA or (HEA); and fo, GAC being likewife = HAE [c] and AC = AE [b], thence will AG = AH [e] and GC = HE.

[b] Conftr.
[c] 3. 1.
[d] Ax. 10. 1.
[e] 15. 1.

Again, fince the right angled triangles CAB, DAB, EAB, FAB have their bafes all equal [f], and the perpendicular AB common, their hypothenufes BC [d], BD, BE, BF will be equal too; and therefore, the triangles CBF, DBE being mutually equilateral, the angle FCB (or GCB) muft be = DEB (or HEB [g]); whence, GC being alfo = HE, and BC = BE, thence is BG = BH [d]: Therefore, AG being likewife (as is proved above) = AH, and AB common, the angles GAB, HAB are equal, and confequently right-angles [h]. In the fame manner, AB is perpendicular to every other ri ht line drawn thro' A in the plane CDEF; *which was to be demonftrated* [i].

[f] Hyp.
[g] 14. 1.
[h] Def. 8. of 1.
[i] Def. 1. of 7.

COROLLARY.

Hence it will appear, that, if one right-line (AB), meeting feveral others (AF, AE, *&c.*) in the fame point (A), is perpendicular to them all; thefe laft will be all in the fame plane. Becaufe it is impoffible for a right line (A*b*) drawn from A, out of the plane (FEDC) of the two former of thefe, to be perpendicular to AB; feeing the angle BA*b* is lefs, or greater, than a right-angle (or BAH [k]), according as A*b* is pofited above, or below the faid plane FEDC [l].

[k] 2. 7.
[l] Ax. 2. 1.

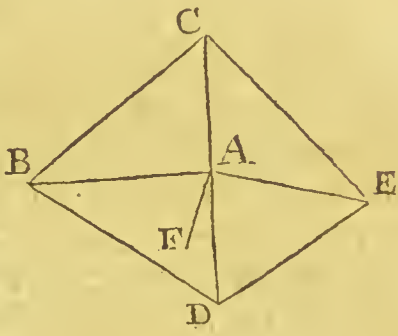
THEOREM III.

If thro' any given point (A) *in a given plane* (BCD), *a line* (CD) *be drawn, and perpendicular to that line, at the fame point* (A), *two other lines* (AB, AE) *be alfo drawn, the one* (AB) *in the plane given* (BCD), *and the other in any other plane* (CDE)

 paffing

passing by the first line (CD); *then, I say, that a right-line* (AF) *drawn from the given point* (A), *at right angles to the first perpendicular* (AB) *in the plane* (BAE *of the two, will be perpendicular to the given plane* (BCD) *at the given point* (A).

For CA being perpendicular both to AB and
m Hyp. AE[m], it will likewise be
n 2. 7. perpendicular to AF[n]; and so FA, being perpendicular to AB[m] (as well as to CA) is also perpendicular to the plane BCD, in which AB and CA are drawn[n].



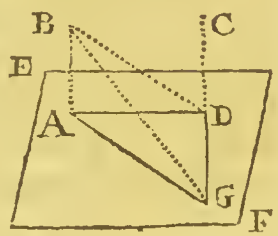
SCHOLIUM.

In this last Theorem, the manner of erecting a perpendicular to a plane, at a point given, is indicated, and the consistence of the *first definition*, of this book, evinced.

THEOREM IV.

Two right-lines (AB, CD), *perpendicular to the same plane* (EF), *are parallel to each other.*

Draw in the plane EF, the right-line AD, and also DG perpendicular to AD; make DG = AB, and let AG, BG, and BD be drawn.



The triangles BAD, ADG,
o Constr. having AB = DG[o], AD com-
p Def.1.7. mon, and the angle BAD (= right-angle[p]) =
q Ax.10.1. ADG[o], will also have BD = AG[q]: And so, the
r 14. 1. triangles BDG, BAG being mutually equilateral, the angle[r] BDG must be = BAG = [p] right-angle: But the line CD (as well as BD and AD) being perpendicular

pendicular to DG[p], it is therefore in the same plane (CDAB) with BD and AD[t]; and consequently, as the angles BAD, CDA are both right-ones[p], it must be also parallel to BA[u].

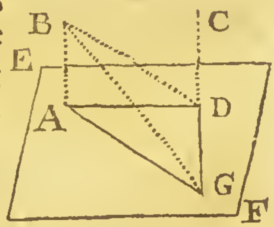
[p] Def. 1. 7. [t] Cor. to 2. 7. [u] 4. 1.

COROLLARY.

Hence it follows, that from the same given point, to one and the same plane, more than one perpendicular right-line cannot possibly be drawn: Because all perpendiculars to the same plane, are parallels; but lines drawn from the same point are not parallels.

THEOREM V.

If, of two parallel right-lines (AB, CD) *the one* (AB) *is perpendicular to any plane* (EF), *the other* (CD) *shall also be perpendiculnr to the same plane* (EF).

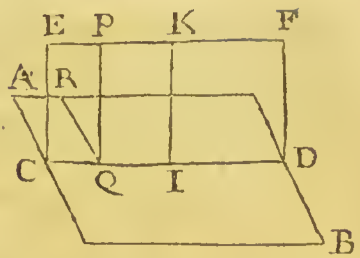


The construction of DG, AG, &c. being supposed the same here as in the preceding Theorem; it appears from thence, that ADG and BDG are both right-angles: And, because BD, as well as AD, is in the plane of the proposed parallels BA, CD[w], the angle CDG is also a right-one[x], as is likewise the angle CDA[y]. Therefore CD is perpendicular to the plane ADG[z].

[w] Def. 6. 1. [x] 2. 7. [y] 4. 1. [z] 2. 7. and Def. 1.

THEOREM VI.

If a right-line (PQ) *be perpendicular to a plane* (AB), *any plane* (ED) *passing by that line, will be perpendicular to the same plane* (AB).



In the plane ED, from any point K, draw KI perpendicular to the common ſection CD. Then, the angle DIK being a right-angle = DQP [y], IK will be parallel to PQ [z], and therefore perpendicular to the plane AB [a]. By the ſame inference, all other right-lines drawn in the plane ED, perpendicular to the common ſection CD, are alſo perpendicular to the plane AB. Therefore the plane ED itſelf is perpendicular to the plane AB [b].

[y] Def. 1. 7.
[z] 4. 1.
[a] 5. 7.
[b] Def. 2. 7.

COROLLARY I.

Hence it will appear, that the plane AB (according to the ſenſe of the definition) is perpendicular to the plane ED: For a right-line QR drawn in the former, perpendicular to the common ſection CD, being alſo [c] perpendicular to PQ, it [d] (and conſequently the plane AB in which it is) will be perpendicular to the plane ED [e].

[c] Hyp. and Def. 1. 7.
[d] 2. 7.
[e] 6. 7.

COROLLARY II.

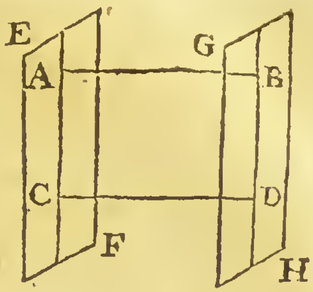
Hence it alſo appears, that a line ſtanding at right-angles to one of two perpendicular planes at any point (I) in the common ſection (CD), muſt be in the other plane: For the line IK, in the plane ED, is perpendicular to the plane AB; beſides which line, another perpendicular to AB, from the ſame point I, cannot be drawn [f].

[f] Cor. to 4. 7.

THEOREM VII.

Planes (EF, GH) *to which one and the ſame right-line* (AB) *is perpendicular, are parallel to each other.*

From



From any point C, in the plane EF, let CD be drawn parallel to AB; which (as well as AB) will be perpendicular to both the planes[g]; and so the angles A, B, C, D (when AC and BD are joined) being all right-ones[h], the figure ABDC (whereof the sides AC, BD are in the same plane with the parallels AB, CD[i]) will therefore be a rectangular parallelogram[k]; and consequently CD = AB[l]. By the very same argument, all other perpendiculars, terminated by the two planes, are equal among themselves; *which was to be demonstrated*[m].

[g] 5. 7.
[h] Def. 1. 7.
[i] Def. 6. 1.
[k] 4. 1. and Def. 24.
[l] 24. 1.
[m] Def. 3.

COROLLARY.

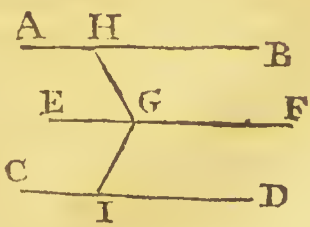
Hence, all right-lines perpendicular to one of two parallel planes, are also perpendicular to the other.

SCHOLIUM.

From the two last Theorems, the sense, and propriety of the two definitions of perpendicular, and parallel planes, appear manifest.

THEOREM VIII.

Right-lines (AB, CD) *parallel to one and the same right-line* (EF), *tho' not in the same plane with it, are also parallel to each other.*



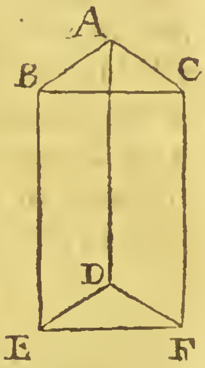
Let GH and GI be drawn perpendicular to EF, in the planes AF and ED of the proposed parallels. Then[n] shall GF be perpendicular to the plane passing by HGI; and HB, ID will also be perpendicular to the same plane[o], and therefore parallel to each other[p].

[n] 2. 7.
[o] 5. 7.
[p] 4. 7.

THEO-

THEOREM IX.

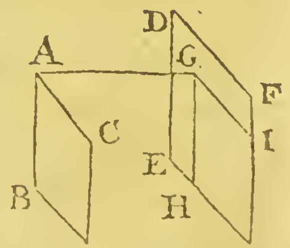
If two right-lines (AB, AC) *meeting each other, be reſpectively parallel to two other right-lines* (DE, DF) *alſo meeting each other, and not being in the ſame plane with them; the angles* (BAC, EDF) *contained by thoſe lines, will be equal.*



Take AB, AC, DE, DF all equal to each other, and let BE, AD, CF, BC, EF be drawn. Then, AB and ED, as well as AC and DF, being
[q] Hyp. and Conſtr. equal, and parallel [q], BE and CF will
[r] 26. 1. be both equal, and parallel, to AD [r], and therefore equal, and parallel, to
[s] Ax. 1. and 8.7. each other [s]; whence BC is alſo equal to EF [r]; and ſo, the triangles ABC, DEF being mutually equilateral, the
[t] 14. 1. angles BAC, EDF are likewiſe equal [t].

THEOREM X.

If two right-lines (AB, AC) *meeting each other, be reſpectively parallel to two other right-lines* (DE, DF) *alſo meeting each other, and not being in the ſame plane with them, the planes* (BAC, EDF) *extended by thoſe lines, will be parallels.*

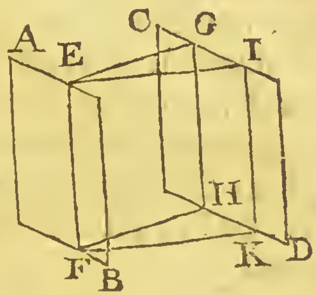


Let AG be perpendicular to the plane BAC, meeting the plane EDF in G; in which laſt plane, let GH and GI be drawn parallel to ED and DF; and they will alſo be parallel to AB
[u] 8. 7. and AC [u]; whence, ſeeing the
[w] Conſtr. angles GAB and GAC are both right [w], AGH and
[x] 5. 1. AGI muſt likewiſe be right-angles [x]; and ſo AG being

being perpendicular to the plane EDF [a] (as well as to BAC [b]), the two planes are parallel to each other [c]. [a] 2. 7. [b] Conftr. [c] 7. 7.

THEOREM XI.

The fections (EF, GH) *made by a plane* (EFHG) *cutting two parallel planes* (AB, CD), *are alfo parallel, the one to the other.*



Let EG and FH be drawn parallel to each other, in the plane EFHG; alfo let EI, FK be perpendicular to the plane CD, and let IG, KH be joined: Then, EG being parallel to FH [d], and EI to FK [e], the angle GEI is = HFK [f]; but the angle EIG is alfo = FKH, being both right-angles [g]; and EI is = FK [h]: Therefore EG will be equal [i] (as well as parallel) to FH; and confequently EF likewife parallel to GH [k]. [d] Conftr. [e] 4. 7. [f] 9. 7. [g] Def. 1. 7. [h] Def. 3. 7. [i] 15. 1. [k] 26. 1.

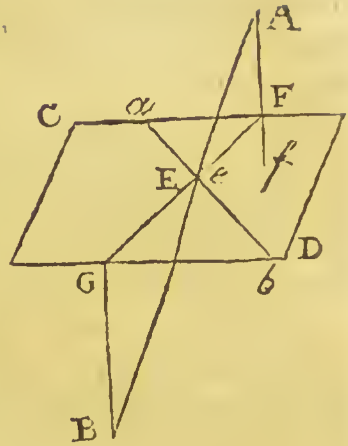
COROLLARY.

It appears from hence, that parallel lines, terminated by the fame parallel planes, are equal to each other.

THEOREM XII.

If, from the two extremes of a right-line (AB) *cutting a plane* (CD), *two perpendiculars* (AF, BG) *be drawn to the plane; the right-line* (FG) *joining the points where they meet the plane, will pafs thro' the point* (E) *in which the propofed line* (AB) *cuts the plane, fo as to be divided by it into two parts* (FE, EG), *having the fame ratio to each other as thofe two perpendiculars* (AF, BG).

For,



For, if AF be produced to *f*, the lines A*f* and BG (which are both perpendicu-
l Hyp. lar to the plane CD[l]) will
m 4. 7. be parallel to each other[m]; therefore AB and FG being both in the same plane with these parallels, in which their
n Def.6.1. extremes are posited[n], they must necessarily (as they are not themselves parallels) intersect each other; And so
o 7. 1. the alternate angles FAE, GBE being equal[o], as
p 3. 1. well as the opposite ones FEA, GEB[p], thence will
q 14. 4. FE : EG : : AF : BG[q]; *which was to be demonstrated.*

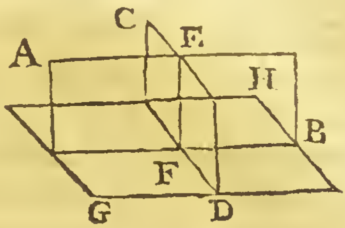
COROLLARY.

Hence, if in the plane CD, the lines FC, GD be made parallel, the one to the other, and in them be taken F*a*=FA, and G*b*=GB; then will the line (*ab*) joining the points *a* and *b*, cut FG in the very same point in which it is cut by AB. For, if *e* be taken as the intersection of *ab* and FG, the trian-
r 3. and gles *a*F*e*, G*eb* will be equiangular[r]; whence F*e* :
7. of 1. *e*G : : F*a* (FA) : [s]G*b* (GB) : : FE : EG[t]. There-
s 14 4. fore, seeing FG is divided in one and the same ra-
t 12. 7.
u Ax. 2. tio, both by *e* and E, these points must necessarily
and 5. coincide[u].
of 4.

THEOREM XIII.

If two planes (AB, CD) *cutting each other, be both perpendicular to a third plane* (GH), *their common section will also be perpendicular to the same plane* (GH).

For,



For, from the extreme point F of the common section, let the right-line FE be erected perpendicular to the plane GH: which line being in both the planes AB, CD [l], it must necessarily be their common section. Therefore the common section is perpendicular to the plane GH [m].

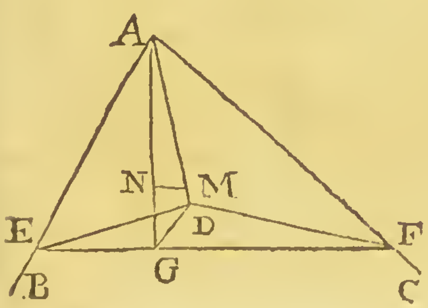
[l] Cor. 2. to 6. 7.

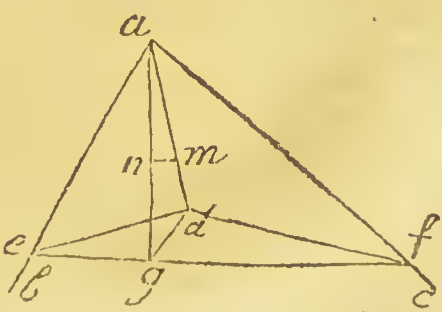
[m] Constr.

THEOREM XIV.

If, from the angular points (A, a) *of two equal angles* (BAC, bac) *two right-lines* (AD, ad) *be drawn, or elevated on high, above the planes of the said angles, so as to form equal angles with the lines first given, each to its correspondent* (DAB = dab, DAC = dac), *and if, from any points* (M, m) *in those elevated lines, perpendiculars* (MN, mn) *be let fall upon the planes* (BAC, bac) *of the first mention'd angles; these perpendiculars will be, in proportion, as the parts* (AM, am) *of the elevated lines included between them and the angular points* (A, a) *first named.*

Make AD and *ad* equal to each other; and in the planes ADB, ADC, *adb*, *adc*, draw DE, DF, *de*, *df* perpendicular to AD and *ad*; and from their intersections with AB, AC, *ab*, *ac*, draw EF and *ef*, meeting AN and *an* (produced) in G and *g*, and let D, G, and *d*, *g* be joined.





The

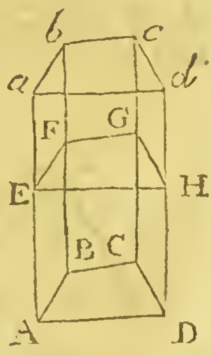
n Conſtr. The angles ADE, ADF being both right-ones [n],
o 2. 7. not only the line AD [o], but the plane ADG ex-
p 6. 7. tended by it, is perpendicular to the plane EDF [p]. But the ſame plane ADG is alſo perpendicular to the plane EAF [p]: Therefore the common ſection EF is likewiſe perpendicular to the plane ADG [q]; and
q 13. 7. r Def. 1. 7. conſequently the angle EGA a right-one [r]. By the very ſame argument, *ega* is a right-angle. Now the triangles ADE, *ade*; ADF, *adf* being equal in all
s Hyp. and 15. 1. reſpects [s], and the angle EAF = *eaf* [t], the triangles AEF, *aef* are alſo equal and alike [u]; and ſo, the
t Hyp. u Ax. 10. 1. angle AEG being = *aeg*, EGA = *ega*, and AE =
w 15. 1. *ae*, thence is AG = *ag* [w], and the angle DAG
x 16. 1. (MAN) = [x] *dag* (*man*), becauſe ADG, *adg* are both
y 14. 4. right-angles [o]. Therefore MN : *mn* : : AM : *am* [y].

COROLLARY.

Hence the two perpendiculars MN, *mn* ſubtend equal angles at the points (A, *a*) from whence the two elevated (or inclining) lines are drawn.

THEOREM XV.

If any ſolid (Ac), *having a rectilinear baſe* (ABCD), *whereof the planes* (Ab, Bc, Cd, Ad) *of the ſides are parallelograms, be cut by a plane parallel to the baſe, the ſection* (EFGH) *will be equal, and ſimilar to the baſe.*



For, the plane EFGH being paral-
z Hyp. lel to ABCD [z], EF is therefore pa-
a 11. 7. rallel to AB [a]; and ſo, AF being a
b Def. 24. of 1. parallelogram [b], EF is equal (as well
c 24. 1. as parallel) to AB [c]. In the ſame manner is FG equal, and parallel to FG, *&c.* Whence alſo the angle
d 9. 7. EFG is = the angle ABC [d]; and ſo of the reſt. Therefore EFGH is both equilateral and equiangular to ABCD.

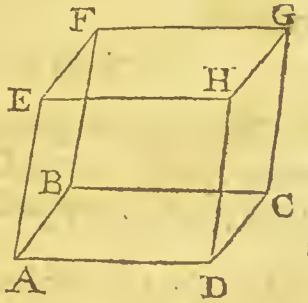
COROL-

COROLLARY.

Hence the opposite bases of a prism are equal and similar (as well as parallel) to each other.

THEOREM XVI.

If, from one of the angular points (A) *of any parallelogram* (AC) *a right-line* (AE) *be elevated above the plane of the parallelogram, so as to make any angles* (EAB, EAD) *with the two contiguous sides thereof, and there be also drawn, from the three remaining angular points, three other right-lines* (BF, CG, DH) *parallel, and equal to the former* (AE); *then, the extremes of those lines being joined, I say, the figure* (AG) *thus described, will be a parallelepipedon.*



For AE, BF, CG, DH being all parallel to each other[c], AE and BF are in the same plane[f], as are also AE and DH, &c. Therefore, all these lines being[g] equal among themselves, AF, AH, DG, and BG are parallelograms[h]; and so, EF being parallel to AB, parallel to DC, parallel to HG[i], EF and HG are in the same plane[f]; and EG is also a parallelogram[k], equilateral to its opposite AC: but EG is equiangular, and parallel (as well as equilateral) to its opposite AC; because, EF being parallel to AB, and EH to AD, the angle FEH is therefore = BAD[l], and the plane EG parallel to the plane AC[m]. And in the same manner, the other opposite parallelograms appear to be equiangular and parallel (as well as equilateral). Therefore the solid AG, bounded by them, is a parallelepipedon[n].

[c] Hyp. and 8. 7.
[f] Def. 13. 1.
[g] Hyp.
[h] 26. 1.
[i] 8. 7.
[k] 25. 1.
[l] 9. 7.
[m] 10. 7.
[n] Def. 7. of 7.

COROL-

COROLLARY.

If the angle A of the parallelogram AC be a right-one, and AE be erected perpendicular to the plane AC; then will the parallelepipedon be a rectangular one: for, all the three contiguous planes AC, AF, AH being rectangular [o], their opposites will be rectangular likewise [p]: and so, the angles HGF, HGC being right-ones, HG will be perpendicular to the plane GB [q]; and consequently both the planes EG and DG likewise perpendicular to the plane BG [r]. And so of the rest.

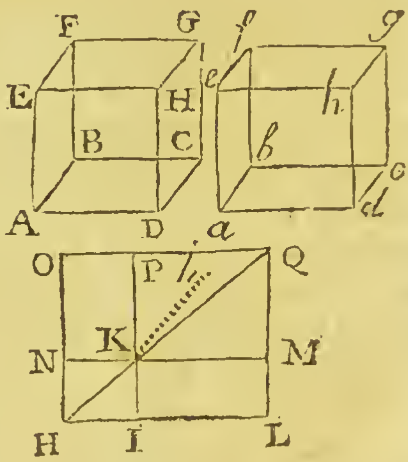
[o] Hyp. and 24. 1.
[p] 16. 7.
[q] 2. 7.
[r] 6. 7.

SCHOLIUM.

In this Theorem, a way to describe a parallelepipedon of any given dimensions, is indicated; and the consistence of the 7th and 9th definitions evinced.

THEOREM XVII.

Rectangular parallelepipedons (AG, ag) *standing upon equal bases* (AC, ac), *and having equal altitudes* (AE, ae), *are equal.*



Let the rectangles OK and KL, equal and like to the bases AC and *ac* of the two solids, be so formed, that NK may be in the same strait line with KM; then shall PK be also in the same strait line with KI [s]; and the figures NI, PM, OL, formed by producing the sides of the two rectangles, will likewise be rectangles [t].

[s] 2. 1.
[t] Cor. to 24. 1.

Now

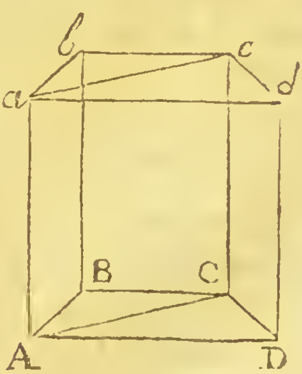
Now, HK and QK being drawn, the triangle NHK = IHK [u], PKQ = MKQ [u], and the rectangle OK = LK [w]; and consequently (by the addition of equals) OQKH = LQKH. Therefore, HKQ being a diagonal to the rectangle OL [x], dividing it into two equal, and like triangles OQH, LQH [u], if upon these, as bases, two upright prisms be conceived to be erected, of the same common altitude (K*b*) with the proposed solids, these prisms will also be equal [y]. But the former of these is composed [z] of three prisms, on the bases OPKN, NHK, KPQ; and the latter of three others, on KMLI, HIK, KMQ; whereof the second and third, in both ranks, are respectively equal [y]. Therefore the remaining two, on the bases OPKN, KMLI must also be [1] equal. But the former of these is = AG [y], and the latter = *ag*: Therefore, also, is AG = *ag*. [a]

[u] 24. 1.
[w] Hyp.
[x] 24. 1. and Ax. 2.
[y] Ax. 1. 7.
[z] Cor. to 18. 7.
[1] Ax. 5.
[a] Ax. 1.

THEOREM XVIII.

If, at the angular points of any given right-lined figure (ABCD), *equal perpendiculars* (Aa, Bb, Cc, Dd) *be erected to the plane thereof, and the extremes of these* (a, b; b, c *&c.*) *be joined; an upright prism* (Aabcd DCBA) *on the given base* (ABCD) *will thereby be formed.*

For, A*a*, B*b*, C*c*, D*d* being all equal [b], and parallel [c], it is evident that A*ab*B, B*bc*C, *&c.* are parallelograms [d]; and that the planes of these are all perpendicular to that of the base ABCD [e] (since A*a*, B*b*, C*c* are so, by construction). Moreover it will appear, that *abcd* is one plane figure, parallel to ABCD;



[b] Constr.
[c] 4. 7.
[d] 26. 1.
[e] 6. 7.

ABCD; for, the lines *ac*, AC (when *a*, *c* and A, C
[f] 26. 1. are joined, being parallels[f] (as well as *ab*, AB; *ad*, AD), the plane *abc* is, therefore, parallel to
[g] 10. 7. ABC[g] (or ABCD); and *acd* is likewise parallel to ABCD. But *abc* and *acd* are in one plane; be-
[h] Cor. to 7. 7. cause A*a* being perpendicular to both of them[h],
[i] Def. 1. 7. and[i] consequently to all the lines *ab*, *ac*, *ad*; these
[k] Cor. to 2. 7. must necessarily be all in one[k] plane, parallel to ABCD; *which was to be demonstrated*[l].
[l] Def. 6.

COROLLARY.

It appears from hence, that, if upon all the parts ABC, ACD, into which any rectilinear figure ABCD is divided, upright prisms (A*abc*CB, A*acd*DC) of the same altitude be constituted; these prisms will form one prism, on the (whole) given base ABCD; seeing that *abc* and *acd* form one continued plane
[m] 18. 7. superficies *abcd*[m], parallel to ABCD.

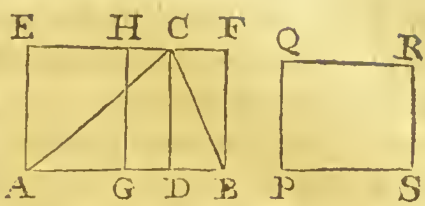
SCHOLIUM.

After the same way, a prism, any how inclining on the given base ABCD, may be constructed; by giving to A*a* the proposed inclination, and then drawing B*b*, C*c*, D*d* parallel, and equal thereto.
[f] 26. 1. For A*ab*B, B*bc*C, &c. will (still) be parallelograms[f]: And, that *abcd* is one plane, parallel to ABCD, will also appear (*in the same manner*); if a perpendicular from *a* to the plane ABCD, be conceived to be drawn.

THEOREM XIX.

If, on equal bases (ABC, PQRS), *an upright triangular prism, and a rectangular parallelepipedon be erected, of the same altitude; the two solids, themselves, will be equal.*

Let

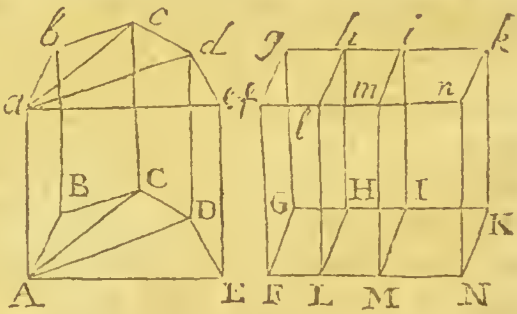


Let CD be perpendicular to AB, and let the rectangles ADCE, BDCF be completed; also let GH be drawn parallel to AE, bisecting AB in G; so shall AGHE = ½ABFE [b] = [b] ACB [c] = PQRS [d]. Now, the two prisms on ADC [c] and BDC, into which, *that* on ABC may be divided [e], will be respectively equal to two others, on the equal and similar [f] bases AEC and BFC [g]: and consequently the prism on ACB = half the prism on AEFB [h] = half the two prisms on AGHE and BGHF [i] = the prism on AGHE = the prism on [k] PQRS *.

[b] 1. 2.
[c] Cor. to 2. 2.
[d] Hyp.
[e] Cor. to 18. 7.
[f] Ax. 24. 1.
[g] Ax. 1. 7.
[h] Ax. 4. 1.
[i] Ax. 3. 1.
[k] 17. 7.

THEOREM XX.

Every upright prism (AaceA) *is equal to a rectangular parallelepipedon* (Fk) *of equal base, and altitude.*



Let AC, *ac*, AD, *ad* be drawn; and in the base FK of the parallelepipedon, let HL, IM, be drawn parallel to FG, in such sort that the rectangles FH, HM, IN may be respectively equal to the triangles ABC, ACD, ADE [k]: Then also shall the prism (A*abc*CBA) [k]

[k] 6. 6.

* *In this Theorem, the representations of the prisms are not described; because a great multiplicity of lines, tends to produce confusion in the mind of a learner; especially where solids are represented. The schemes, however, may be formed, at large, by those who think proper to do it: But every tittle of the demonstration will remain the same.*

on ABC, be equal to the parallelepipedon (F*b*) on
[l] 19. 7. FH [l]; and the prifm (A*acd*DCA) on ACD, equal to the parallelepipedon (L*i*) on LI [l]; and confequently the whole prifm (A*ace*A) on ABCDE, equal to all the parallelepipedons on FK, which form one parallelepipedon (F*k*); becaufe L*l*, M*m* are
[m] 18. 7. and Cor. 2. to 6. 7. in the plane F*n* [m], and H*h*, I*i* in the plane G*k* [m].

COROLLARY.

Hence all upright prifms, having equal bafes, and altitudes, are equal among themfelves.

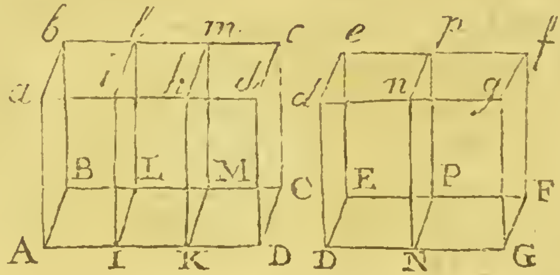
SCHOLIUM.

In the very fame manner, the aggregate of any number of prifms, of one common altitude, will appear to be equal to one fingle prifm, or parallelepipedon, of the fame altitude, whofe bafe is equal to the fum of all theirs.

THEOREM XXI.

Rectangular parallelepipedons (Ac, Df) having equal altitudes (Aa, Dd) are in the fame proportion as their bafes (AC, DF).

Let the proportion of the bafe AC to the bafe DF, be that of any one number *m* (3) to any other number *n* (2).



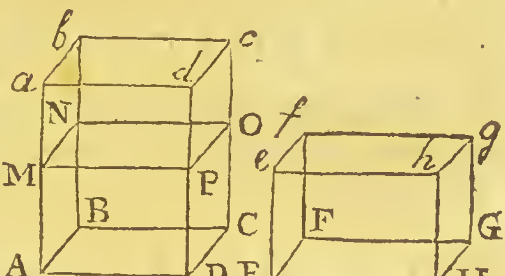
Let AC be divided into *m* (3) equal parts (or rectangles) AL, IM, KC (by dividing AD into that
[m] 11. 5. number of equal parts [m], and drawing IL, KM pa-
[n] 9, 5. and 1. 2. rallel to AB) [n]: And let DF be divided, in like manner, into *n* (2) equal parts, or rectangles, DP, NF: Which parts, taken fingly, will be equal, in
[o] Hyp. and Ax. 8. 4. magnitude, to thofe of the former divifion [o]; and fo the

the parallelepipedons upon them (A*l*, I*m*, K*c*, D*p*, N*f*) will likewise be all equal[p]: Therefore the solid [p] 17. 7.
A*c* is in proportion to the solid D*f*, as the number of parts in A*c* to the number of equal parts in D*f*[q], or [q] Ax. 8.7.
as the number of parts in AC to the number of equal parts in DF, that is, as AC to DF[q].—If the bases are supposed to be *incommensurable*, the solids will still be in the same ratio with them, as appears from the reasoning laid down in the *Scholium* to *Theor* VII. *Book* IV.; which is equally applicable in this case.

THEOREM XXII.

Rectangular parallelepipedons (A*c*, E*g*) *standing upon equal bases* (AC, EG) *have the same ratio as their altitudes* (Aa, Ee).

Let AO be a parallelepipedon on the base AC, whereof the altitude AM is equal to that (E*e*) of the parallelepipedon E*g*: So shall



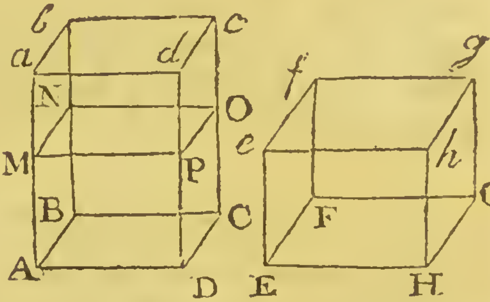
the solid AO = the solid E*g*[s]. (But if A*b* and [s] 17. 7.
AN be considered as bases) it will be A*c* : AO (or E*g*) : : A*b* : AN[t] : : A*a* : AM[u] (or E*e*) : *which was* [t] 21. 7.
to be proved. [u] 7. 4.

COROLLARY.

Hence, and from the preceding Theorem, it follows, that all upright prisms are, also, as the bases, when the altitudes are equal; and as the altitudes, when the bases are equal; all such solids being (*by Theor.* XX.) equal to rectangular parallelepipedons of equal base and altitude.

THEOREM XXIII.

Upright prisms and parallelepipedons (A*c*, Eg) *which have their bases and altitudes reciprocally proportional* (AC : EG : : Ee : Aa), *are equal to each other.*



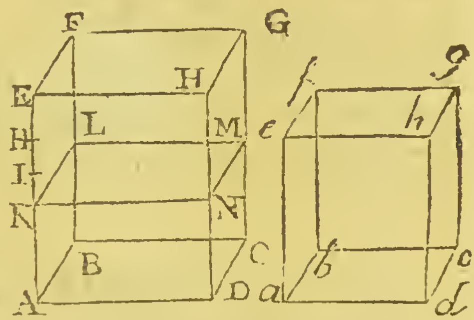
Let AO be a prism on the base AC, whereof the altitude AM is equal to that (F*e*) of the prism Eg.

Then AO : Eg : : AC : EG[w] : : E*e* (AM) :[x] A*a* : : AO :[y] A*c*; and consequently Eg = A*c*[z].

[w] 21. 7.
[x] Hyp.
[y] 22. 7.
[z] Ax. 4. of 4.

THEOREM XXIV.

Similar upright prisms and parallelepipedons (AG, ag) *are, to one another, in the triplicate ratio of their altitudes* (AE, ae.)



Having made AH = *ae*, take AE, AH, AI, AK in continued proportion[a]; and let AM be a prism on the base AC, whereof the altitude is AK. Then (because of the similar planes) it will be AC : *ac* : :[b] AB^2 : *ab*^{2} : :[c] AE^2 : *ae*^{2} (AH^2) : : AH^2 :[c] AI^2 : : AH (*ae*) : AK[c]; and so, the bases and altitudes of the solids AM, *ag* being reciprocally proportional, the solids themselves are equal;

[a] 13. 5.
[b] 26. 4.
[c] Cor. to 11. 4.
[e] 27. 4.

equal [f]; and therefore, AG : *ag* : : AG : AM [g] : : AE : AK [h] : But the ratio of AE to AK is triplicate to that of AE to [i] AH (or *ae*). Therefore, &c.

[f] 23. 7.
[g] Ax. 1. of 4.
[h] 22. 7.
[i] Def. 7. of 4.

COROLLARY I.

Hence, cubes are in the triplicate ratio of their ſides, or altitudes.

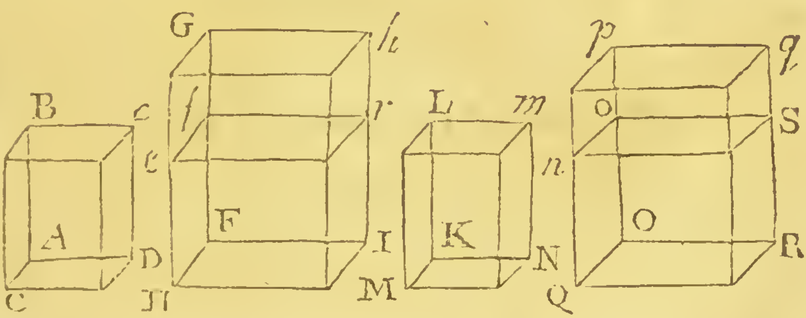
COROLLARY II.

Hence, alſo, all ſimilar upright priſms, are to one another, as the cubes of their altitudes ; ſince both priſms and cubes, are in the *ſame* triplicate ratio of the altitudes.

THEOREM XXV.

Rectangular parallelepipedons, contained under the correſponding lines of three ranks of proportionals, are themſelves proportionals.

$$\textit{I say, if}\ \begin{cases} AB : FG :: KL : OP, \\ AC : HF :: MK : QO, \\ AD : FI :: KN : OR, \end{cases}$$



then ſhall the ſolid (Cc) *contained under the three firſt antecedents, be to that* (Hh) *contained under their three conſequents ; as the ſolid* (Mm) *contained under the three other antecedents, is to that* (Qq) *contained under the three remaining conſequents.*

 Let

Let Hr and Qs be parallelepipedons on the bafes HI and QR, of the fame altitude with Cc and Mm refpectively.

[k] 21. 7. Then fhall $Cc : Hr :: $ bafe CD : bafe HI[k];

And $Mm : Qs ::$ bafe MN : bafe QR[k]. But the four bafes, becaufe of the proportionality of

[l] 11. 4. their fides, are themfelves proportionals[l]: and fo, by equality, the ratio of Cc to Hr, is the fame as the ratio of Mm to Qs. And the ratio of Hr to Hh is likewife the fame as that of Qs to Qq (be-

[m] 22. 7. caufe $Hr : Hh :: Ff$ (AB) : FG[m] :: KL (Oo) :

[n] Hyp. Op[n] :: $Qs : Qq$[m]). Therefore fhall the ratio of Cc to Hh, be alfo the fame, as the ratio of Mm

[o] 5. 4. to Qq[o].

COROLLARY.

Hence the cubes of four proportional lines are proportional.

The End of the SEVENTH BOOK.

ELEMENTS

OF

GEOMETRY.

BOOK VIII.

POSTULATES.

1. THAT, of any two unequal magnitudes, of the ſame kind, the leſs may be multiplied ſo often, till it exceed the greater.

2. That, a right-line may be taken ſo ſmall, that the ſquare thereof ſhall be leſs than any ſuperficies aſſigned.

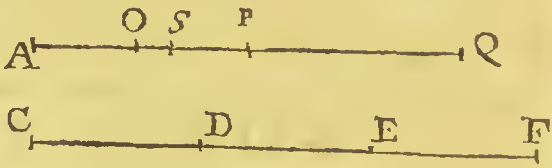
3. That, the circumference of a circle is greater than the perimeter (or the ſum of all the ſides) of any inſcribed polygon; and leſs than the perimeter of any polygon deſcribed about the circle.

What is required to be granted, in the ſecond of theſe three Poſtulates, might be effected and proved, in form, by means of the Firſt; but being itſelf more obvious (if poſſible) than even that, *it ſeemed unneceſſary to make it depend thereon.*

L E M-

LEMMA I.

If from the greater (AQ) *of two unequal magnitudes* (AQ, CD) *there be taken the half* (PQ), *and from the remainder* (AP) *be again taken the half* (PO), *and so on, continually; there shall at length be left a magnitude, less than the least* (CD) *of the two magnitudes first propounded.*



Take DE = CD, EF = CD, and let this be so often done, till the multi-
[a] Post. 1. ple CF exceed AQ [a]. Let the proposed bisections of AQ, AP, &c. be continued till the parts PQ, OP, AO be equal in number to the parts EF, DE,
[b] Hyp. CD. Now AP ($\frac{1}{2}$AQ [b]) $\sqsupset$ $\frac{1}{2}$CF [b] $\sqsupset$ CE. And, in the same manner, AO ($\frac{1}{2}$AP) $\sqsupset$ $\frac{1}{2}$CE (CD); *which was to be done.*

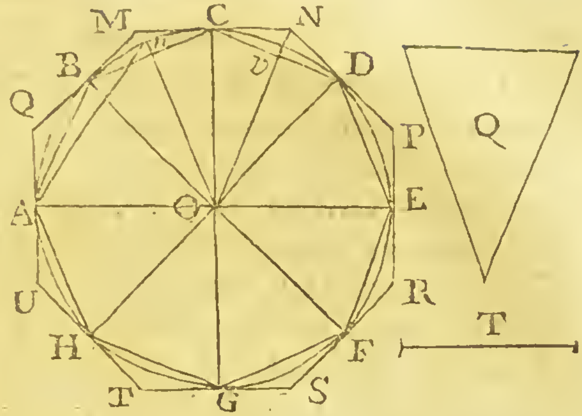
SCHOLIUM.

When the magnitudes given (AQ, CD) are right lines, a part, or measure (AS) of the one, less
[c] 11. 5. than the other, may be found at one operation [c]; by taking AS the same part of AQ, as CD is of CF [c]. For, the whole AQ being less than the whole CF [b], the part AS will also be less than the
[d] Cor. 1.4. part CD [d].

THEOREM I.

Two polygons may be formed, the one in, the other about a given circle, which shall differ less from each other (and consequently from the circle itself) than by any assigned magnitude (Q) however small.

Let



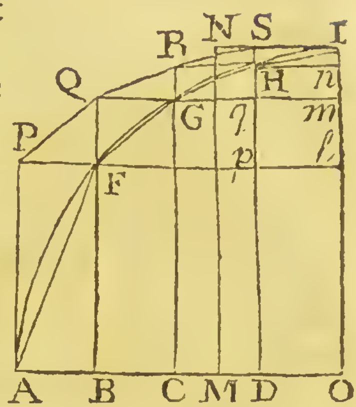
Let T be the side of a square equal to, or less than Q[c]; and in the circle apply $An = T$[f]: and, having drawn the two perpendicular diameters AE, CG, proceed by a continual bisection of the angles at the center, till you arrive at an angle AOB less than the angle A*On* subtended by *An*[g]: Inscribe the regular[h] polygon ABCDEFGH, by making the angles BOC, COD, &c. equal to AOB: and let another regular polygon QMNPRSTU, of the same number of sides, [i] be described about the circle; which will exceed the inscribed one by a magnitude less than Q.

For, if to any angle N of the greater, ON be drawn, it will bisect the same[k], and will cut the side CD of the inscribed polygon at right angles[l] (in *v*): And so, the triangles OCN, *v*CN being equiangular[m], they (and consequently their doubles OCNDO, CND) will be in proportion to each other, as [n] OC^2 to Cv^2, or as [o] AE^2 to AB^2. And it is manifest, that the whole circumscribing polygon (OCND + ODPE &c.) must be to its whole excess (CND + DPE, &c.) above the inscribed one, in the same [o] proportion of AE^2 to AB^2. But the first *antecedent* is less than the *second*, or than a square described about the circle[p]: Therefore the first *consequent* (CND + DPE &c.) is also $\sqsupset$ [q] AB^2 $\sqsupset$ [r] An^2 (T^2) $\sqsupset$ [s] Q.

[c] Post. 2.
[f] 20. 5.
[g] 5. 5. and Lem. 1.
[h] Ax. 10. and 4. 1.
[i] 30. 5.
[k] 16. 1.
[l] Cor. to 12. 1.
[m] 19. 4.
[n] 24. 4.
[o] 1. 4.
[p] Ax. 2.
[q] 2. 4.
[r] 21. 1.
[s] Hyp.

Other-

Otherwise.



Let AIO be one quadrant of the proposed circle; and
t 7. 6. on the radius OI, make [t] the rectangle OMNI $= \frac{1}{4}T^2$, (T being as before): Take OD a part of OA, less than
u Lem. OM [u]; and, having made AB, BC, &c. each $=$ OD, draw AP, BFQ, CGR, DHS
w 3. 5. perpendicular[w] to AO, meeting the circumference in A,
x 9. 5. F, G, H; through which points, parallel to AO [x], draw PF*l*, QG*m*, RH*n*, meeting AP, BFQ, CGR, in P, Q, R: Join PQ, QR, RS, SI, as also AF, FG, GH, HI. Then will the two polygons OAFGHI and OAPQRSI (whereof one is less, and the other greater than the quadrant) differ less from each other, than by $\frac{1}{4}$ of the proposed quantity Q.

For, that the former OAFGHI is less than the
y Ax. 2. quadrant, in which it is inscribed, is manifest [y]: And, that the latter is greater than the quadrant, will also plainly appear; seeing two sides AP, SI,
z 6. 3. only, touch the circumference [z]; all the rest PQ, QR, RS, falling wholly above it, as being sides of triangles PFQ, QGR, RHS formed out of the cir-
a Constr. and Ax. 2. cle [a]. Now the excess of the polygon OAPQRSI above OAFGHI, is composed of the triangle PAF ($= \frac{1}{2}$CD*pl*) and of all the parallelograms PFGQ,
b Ax. 3. QGHR [b], RHIS (for they are such [c], because PF
c 26. 1. (AB), QG (BC) are equal, as well as parallel [d]).
d Constr. and 26. 1. Therefore PFGQ being $=$ *plmq* [e], QGHR $=$
e Cor. 2. to 2. 2. *qmn*H [e], &c. the said excess will consequently be $= \frac{1}{2}$OD*pl* $+$ *lp*SI [b] $\sqsupset$ ODSI [f] $\sqsupset$ OMNI ($\frac{1}{4}T^2$)
f Ax. 2. $\sqsupset \frac{1}{4}$ Q [g]; *which was to be done.*—This last con-
g Hyp. struction is equally applicable to other curvilineal figures; the former is peculiar to the circle.

COROL-

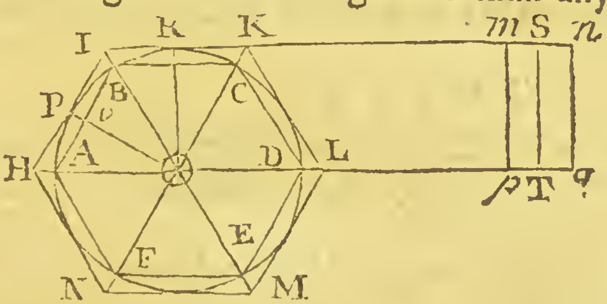
COROLLARY.

It follows from hence, that a magnitude, which is greater than any polygon that can be defcribed in, and lefs than any polygon that can be formed about a given circle, muft be equal to the circle itfelf: feeing that a polygon may be infcribed, which (as well as that formed about the circle) fhall exceed any quantity lefs than the circle itfelf, be the difference ever fo fmall; and becaufe a polygon may be formed about the circle, which (as well as that in the circle) fhall be lefs than any quantity that exceeds the circle.

THEOREM II.

Every circle (ACE) *is equal to a rectangle* (ORST) *under the radius thereof* (OR) *and a right line* (OT) *equal to half the circumference.*

It is evident, in the firft place, that the propofed rectangle ORST is greater than any poly-



gon ABCDEF that can be defcribed in the circle: For, drawing OA, OB, *&c.* And alfo Ov perpendicular to AB; it is plain, that the triangle AOB ([h] Ov × ½ AB) will be lefs than [i] OA × ½AB (or OR×½AB): And, in the fame manner, BOC ⊐ OR × ½BC, *&c.* Confequently, the whole polygon ABCDEF is lefs than [k] OR × ½ AB + OR × ½ BC, *&c.* that is [l], lefs than a rectangle (Om) under (OR and Op = half the perimeter AB + BC + CD *&c.*) But this rectangle (Om) is, itfelf, lefs than

[h] Cor. to 2. 2
[i] 20. 1 and Ax. 2.
[k] Ax. 2.1.
[l] 5. 2.

than OS, becaufe, Op (half the perimeter of the
[m] Poft. 3. polygon) is lefs than OT[m] (half the circumference of the circle). Confequently the polygon ABCDEF is lefs than the rectangle OS.

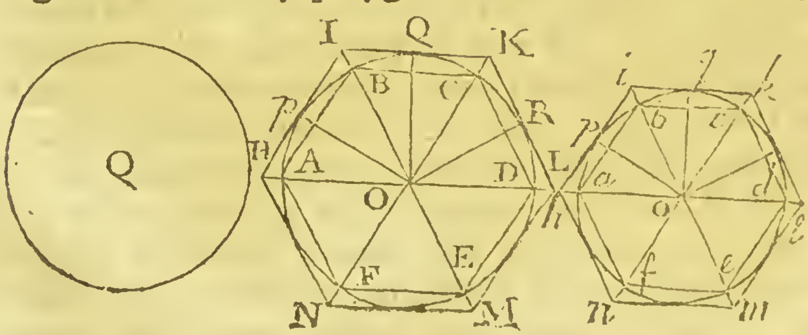
But, fecondly, it will appear that the fame rectangle ORST is lefs than any polygon HIKLMN that can be defcribed about the circle: For, if OH, OI, &c. be joined, and the radius OP be drawn to the point where HI touches the circle; then will
[n] Cor. to 2. 2. the triangle $HOI = {}^{n}OP \times \frac{1}{2}HI$ ($= OR \times \frac{1}{2}HI$). In the very fame manner $IOK = OR \times \frac{1}{2}IK$, &c. and therefore the whole polygon HIKLMN $=$
[o] Ax. 4. 1. [p] 5. 2. ${}^{o}OR \times \frac{1}{2}HI + OR \times \frac{1}{2}IK$, &c. $= {}^{p}$ a rectangle (On) under OR and $Oq =$ half the perimeter ($HI + IK + KL$, &c.); which rectangle is, manifeftly greater than OS, fince Oq ($=$ half the pe-
[q] Poft. 3. rimeter of the polygon) is greater than OT[q].

Seeing therefore, that the rectangle OS is greater than any polygon that can be defcribed in the circle, and lefs than any polygon that can be defcribed
[r] Cor. to 1. 8. about the circle; it muft be equal to the circle[r].

THEOREM III.

All circles (ACE, ace) *are in proportion to one another, as the fquares of their radii* (AO^2, ao^2).

Let Q : circle *ace* : : $AO^2 : ao^2$; then I fay, that Q $=$ circle ACE. For, firft, it is evident that Q is greater than any polygon ABCDEF that can be



defcribed

deſcribed in the circle ACE: Becauſe, if another polygon *abcdef*, ſimilar thereto, be [s] deſcribed in the circle *ace*; then will polyg. ABCDEF : polyg. *abcdef* (: : [t] $AO^2 : ao^2$,) : : Q : [u] circle *ace*; where the firſt *conſequent* (polyg. *abcdef*) being leſs than the *ſecond* (or, than the circle in which it is inſcribed [w]) it is manifeſt, that the firſt *antecedent* ABCDEF muſt alſo be leſs than the ſecond Q [x].

[s] 31. 5. [t] Cor. to 31. 5. [u] Hyp. [w] Ax. 2. [x] 2. 4.

In the ſame manner it will appear, that Q is leſs than any polygon HIKLMN that can poſſibly be deſcribed about the circle ACE: For, if about the other circle *ace*, a ſimilar polygon *hiklmn* be deſcribed [y]; then will HIKLMN : *hiklmn* (: : [z] $AO^2 : ao^2$) : : Q : [a] circle *ace*; where the firſt *conſequent* (*hiklmn*) being greater than the ſecond (*ace*) [b], the firſt *antecedent* HIKLMN muſt therefore be alſo greater than the [c] ſecond Q.

[y] 31. 5. [z] Cor. to 31. 5. [a] Hyp. [b] Ax. 2. [c] 2. 4.

Therefore, ſeeing that Q is greater than any polygon that can be deſcribed in the circle ACE, and leſs than any polygon that can be deſcribed about the circle; it muſt be equal to the circle [d].

[d] Cor. to 1. 8.

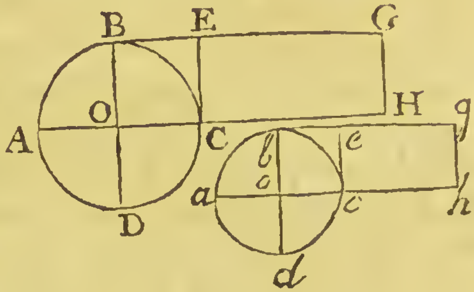
SCHOLIUM.

After the ſame manner, other ſimilar curvilineal figures are proved to be in proportion, as the ſquares of their diameters, or other homologous dimenſions; by means of the ſecond conſtruction of the firſt propoſition; it being very eaſy to demonſtrate, that the polygons formed from thence, whether both within, or both without two ſimilar figures, will themſelves be ſimilar.

THEO-

THEOREM IV.

The circumferences of all circles (ABCD, abcd) *are in the same proportion as their radii* (OB, ob.)

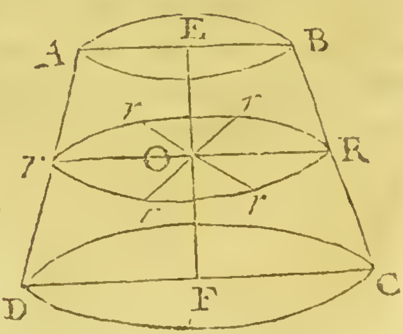


Let OE, *oe* be squares on the radii OB, *ob*; and let OG, *og* be two rectangles contained under the same radii and right lines OH, *oh*, respectively equal to the semi-circumferences ABC, *abc*. Then, these rectangles being equal to the circles themselves[d], it will therefore be, OE : OG : : *oe* : *og*[e]. And in this same ratio are[f] also the bases OC, OH ; *oc*, *oh* : whence (*by equality and alternation*) OC (OB) : *oc* (*ob*) : : OH : *oh* : : 2OH (circumf. ABCD[g]) : 2*oh* (circumf. *abcd*).

[d] 2. 8.
[e] 3. 8.
[f] 7. 4.
[g] Hyp.

LEMMA 2.

If a solid (AC) *generated by the revolution of any plane figure* (EBCF) *about a quiescent axis* (EF), *be cut by a plane perpendicular to the axis; the section will be a circle, having its center in the point* (O) *where it meets the axis.*



For, from O, in the generating plane EBCF, draw OR perpendicular to the axis EF, meeting BC in R.

Then, since this line OR, during the whole revolution, every-where preserves its perpendicu-

larity

larity to the axis EF, it is therefore always in the plane paſſing through O perpendicular to the ſaid axis [h]: and conſequently, as the length thereof alſo continues the ſame in every poſition, the line R*rrrr* deſcribed, in that plane, by the extreme point R, by which the ſection is bounded, muſt be the circumference of a circle [i], whereof the point O is the center.

[h] Cor. to 2. 7.

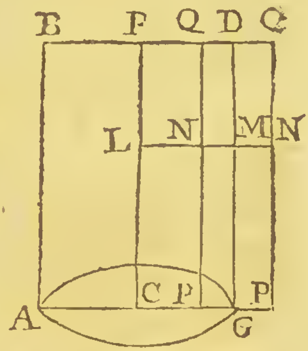
[i] Def. 33. 1.

COROLLARY.

Hence, not only the baſes of cylinders and cones, but all ſections parallel to them, are circles.

LEMMA 3.

A right-line (PQ) *ſtanding perpendicular to the plane of a cylinder's baſe (and not exceeding the axis* CF) *falls wholly within, or wholly without the cylinder, according as the point* (P) *on which it inſiſts, is ſituate within, or without the circumference of the baſe.*



From the center C, to the given point P, draw CP; take, in CF and PQ, any two equal diſtances CL, PN, and let LN be drawn, meeting the ſurface of the cylinder in M.

Becauſe CL and PN are parallel [k], and therefore both in the ſame plane [l], LN is parallel, and equal to CP [m]. Therefore, when CP is leſs than the radius CG, LN will be leſs than CG, or than its equal LM [n]; and ſo the point N muſt fall within the cylinder [o]. And the ſame is equally true with regard to any other point in the line PQ. But, when CP is greater than CG, LN will alſo be greater than HG (LM); and the point N will then fall out of the cylinder [o].

[k] 4. 7.

[l] Def. 13.

[m] 26. 1. 24. 1.

[n] Def.12. and 7.

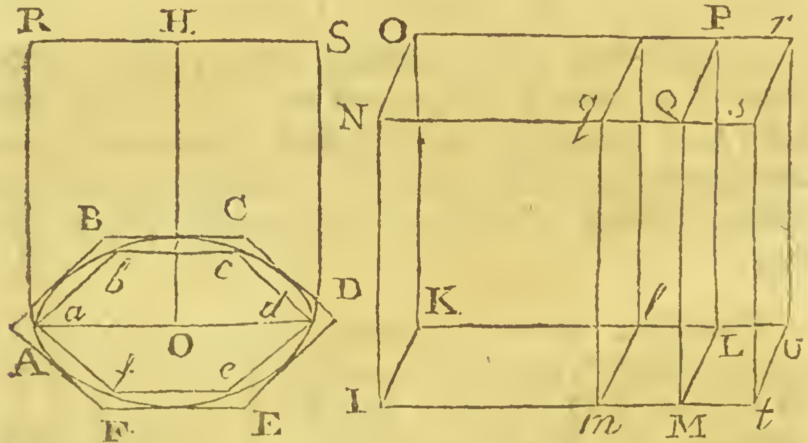
[o] Ax. 2.1.

THEOREM V.

Every cylinder is equal to a rectangular parallelepipedon of equal base and altitude.

I say, if the base *ace* of the cylinder *a*S be equal to the base IKLM of the rectangular parallelepipedon IP, and the altitude OH of the former be also equal to the altitude KO of the latter; then the two solids will be equal.

For, first, it is evident, that the cylinder exceeds any parallelepipedon (I*p*), of the same given altitude, whose base I*klm* is less than the base (*ace*) of



the cylinder. Because a polygon (*abcdef*) may be
p 1. 8. described in the circle *ace*, that shall exceed IK*lm* [p];
upon which, an upright prism (of the given alti-
q 18. 7. tude) may be constituted [q]; which will be less than
the cylinder as being wholly contained therein;
since (*by Lemma* 3.) all right-lines drawn perpen-
r Cor. 2. dicular to the base, in the planes of the sides [r],
to 6. 7. from any points in *ab*, *bc* &c. fall wholly within the
cylinder, and consequently the planes themselves,
in which they are. But this contained prism is
s 21. 7. greater than the parallelepipedon I*p* [s]: therefore
the cylinder itself must, necessarily, be greater
t Ax. 2. than I*p* [t],

In

In like manner it will appear, that the cylinder is leſs than any parallelepipedon I*r* (of the ſame altitude) whoſe baſe IK*vt* exceeds that of the cylinder: For a polygon (ABCDEF) may be deſcribed about the circle *ace* that ſhall be leſs than IK*vt* [u]; upon which a priſm may be conſtituted [w], which, tho' leſs than I*r* [x], will, nevertheleſs, exceed the cylinder [y].

[u] 1. 8.
[w] 18. 7.
[x] 21. 7.
[y] Ax. 2.

Therefore, ſeeing that the cylinder can neither be leſs, nor greater than IP; it muſt neceſſarily be equal to it.

COROLLARY.

Hence, whatever is demonſtrated in the 21ſt, 22d, and 23d Theorems of the preceding Book, with reſpect to the proportions of priſms, holds equally true in cylinders alſo; being equal to priſms of equal baſe and altitude [z].

[z] 20. 7. and 5. 8.

SCHOLIUM.

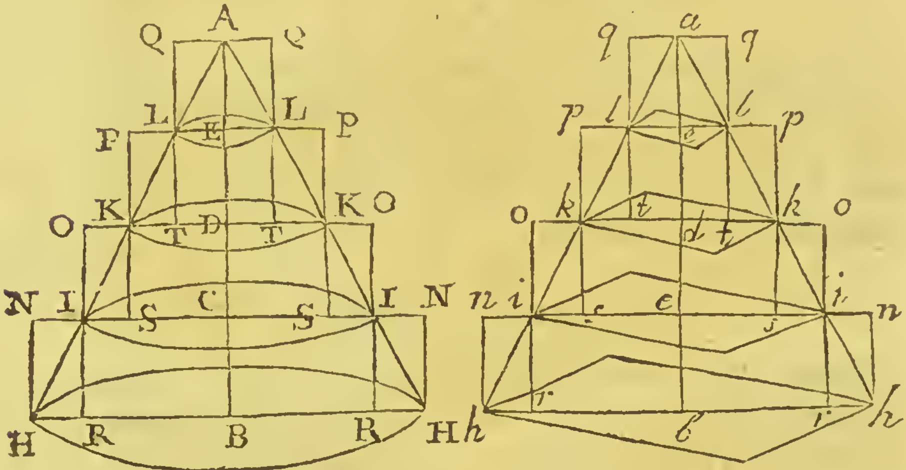
From the ſame demonſtration, it will likewiſe appear, that every regular ſolid, whoſe ſections, by planes perpendicular to the baſe, are all rectangles; is equal to a parallelepipedon of equal baſe and altitude; and conſequently, that all ſolids of this kind (which may be comprehended under the name of *Cylinderoids*) will be equal among themſelves, when their altitudes, as well as baſes, are equal.

LEMMA 4.

If two ſolids (HAH, hah) *of the ſame altitude have their ſections by planes parallel to the baſes, at all equal diſtances therefrom, equal to each other; it is propoſed to demonſtrate (under certain reſtrictions ſpecified hereafter) that the ſolids themſelves will be equal.*

 Let

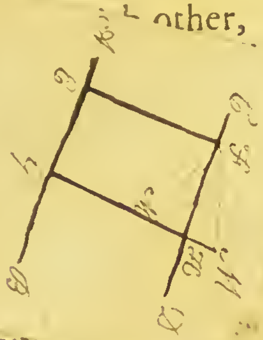
Let II, KK &c. *ii*, *kk*, &c. be sections of the two solids by planes parallel to the bases HH, *hh*, dividing the altitudes AB, *ab* into parts BC, CD &c. *bc*, *cd* &c. all mutually equal to each other. Then,
[m] Hyp. every two corresponding sections being equal[m] (HH = *hh*, II = *ii*, &c.) the upright solids HNNH, *hnnh*; IOOI, *iooi* &c. formed thereon, will also be,
[n] 20. 7. and Sch. 5. 8. respectively, equal one to another[n], whether they be prisms, cylinders, or cylinderoids, that is, whether the sections themselves be right-lined figures, circles, or curvilineal figures of any other kind.



Now if these sections HH, II *&c.* be supposed to decrease, from the base upwards, so that the solids (HNNH, IOOI *&c.*) formed upon them, may exceed the correspondent parts (HIIH, IKKI *&c.*) of the given solid HAH; it is manifest, that the sum of all the said solids (HNNH + IOOI *&c.*)
[o] Ax. 2. will likewise exceed the whole proposed solid HAH[o]. But, if within HAH, on the same sections (but on contrary sides thereof) another series of such solids IRRI, KSSK *&c.* be formed; the sum of all these will, manifestly, be less than the proposed solid
[p] Ax. 4. HAH, in which they are contained[p]. And it is also evident, that this last series will be less than the former (HNNH + IOOI *&c.*) by the greatest of these

theſe ſolids HNNH; becauſe (this one, alone, being excepted) to every other ſolid of the rank, an equal, in the contained rank, may be aſſigned, and *vice verſâ:* For [q] IRRI = IOOI, KSSK = KPPK, LTTL = LQQL. [q] 20 7. and Sch. 5. 8.

Now, ſince the altitude BC, of the ſolid HNNH whereby the contained, and containing ſeries, differ ... other, may be taken ſo ſmall a part of ... itſelf ſhall be leſs than any aſ-



...hatever [r]; it is manifeſt (from [r] Lem. 1. and 22.7.
...ol. to Theor. I.) that a magni-
...ter than any ſeries of ſolids (of
...ified) that can be formed within
HAH, and leſs than any ſe-
formed about HAH, muſt be
equal ... But the ſolid *hab*, being greater than any ſeries of ſolids (*irri*+*kssk* &c.) contained therein [s], is therefore greater than any ſeries of ſolids IRRI+KSSK *&c.* contained in HAH (*theſe*, being, reſpectively, equal to *thoſe*): And the ſame ſolid *hab*, being leſs than any ſeries of ſolids (*hnnh* + *iooi* &c) formed about it, is alſo leſs than any ſeries of ſolids (HNNH+ IOOI *&c.*) that can be formed about HAH. Therefore the ſolid *hab* is equal to HAH. [s] Ax. 2.

In this demonſtration, the ſections are ſuppoſed to decreaſe, continually, from the baſes upwards; ſo as to have the ſides of the upright ſolids formed thereon, placed wholly without, or wholly within, the ſuperficies of the given ſolids HAH, *hab*: Which can only be the caſe, when all perpendiculars, from any points in the ſurface of either, to the plane of the baſe, fall within the limits of the baſe. If, however, the ſections be ſuppoſed to decreaſe to a certain diſtance, only, and then to increaſe again; the two ſolids will, ſtill, appear to be equal: Becauſe the parts of the one, terminated

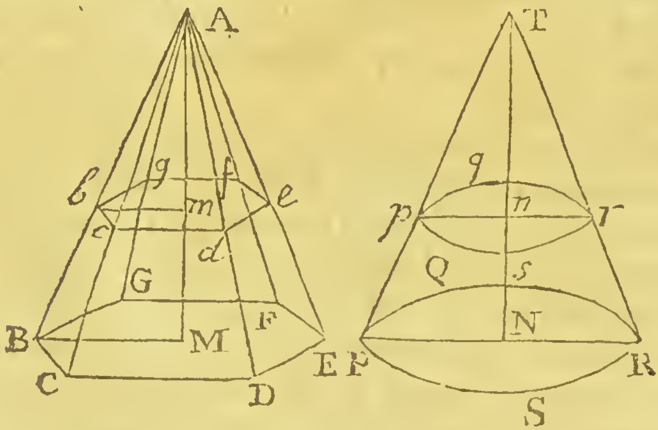
 by

by ſuch limits of decreaſe; or increaſe, will (by the ſame demonſtration) be reſpectively equal to the correſpondent part of the other. But, as no ſuch ſolids have a place in the *Elements of Geometry*, to ſay more about them here, would be improper.

LEMMA 5.

If pyramids and cones (ABCDEFG, TPQRS) *having equal altitudes* (AM, TN), *be cut by planes parallel to the baſes*; *the ſections* (bcdefg, pqrs), *at all equal altitudes* (Mm, Nn), *will be in the ſame proportion as the baſes.*

For, the plane *bcdefg* being parallel to BCDEFG,
[a] 11. 7 thence is *bc* parallel to BC [a], *bg* to BG, &c. and
[b] 9. 7. conſequently the angle *cbg* = CBG [b], *bcd* = BCD,
[c] 14. 4. &c. alſo *bc* : BC (: : A*b* : AB [c]) : : *bg* : BG. And, in the ſame manner, the ſides about the other equal angles are proportional. Therefore, the two po-



[d] Def. 14. 4. lygons *bcdefg*, BCDEFG being ſimilar [d], they are
[e] 26. 4. in proportion [e], as bc^2 to BC^2, or as Ab^2 to [f] AB^2,
[f] Cor. to 11. 4. or, laſtly, as AM^2 to [f] AM^2; becauſe (BM and *bm*
[g] Cor. to 7. 7. being drawn) the angles AMB, *amb* will be right-ones [g], and *bm*, therefore, parallel to BM [h].
[h] 4. 1.

But

But the ſection *pqrs* is *alſo* to the baſe PQRS in the *ſame* proportion of Am^2 (Tn^2) to AM^2 (TN^2); becauſe *pqrs*, PQRS, being [i] circles, they are as the ſquares of their radii *pn*, PN[k], and conſequently as Tn^2 to [l] TN^2. Therefore, ſeeing that the two ſections have both the ſame ratio to their reſpective baſes, the propoſition is manifeſt[m].

[i] Lem. 2.
[k] 3. 8.
[l] Cor. to 11. 4.
[m] 2. 4.

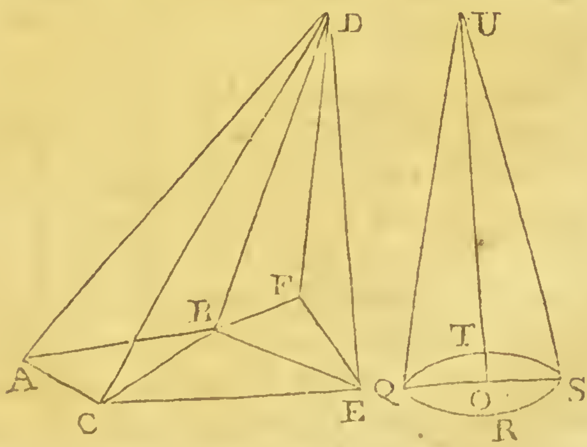
COROLLARY.

It appears from hence, that the ſection of any pyramid, by a plane parallel to the baſe, is ſimilar to the baſe.

THEOREM VI.

All pyramids of the ſame altitude ſtanding upon equal triangular baſes, are equal among themſelves; and every ſuch pyramid (ABCD) *is equal to a cone* (QRSTU) *of equal baſe and altitude.*

Case I. If the perpendicular, let fall from the vertex D of the pyramid upon the plane of the baſe ABC, falls not out of the baſe, or beyond the limits of the triangle: Then it is manifeſt, *from Lemma* 4, ſeeing the ſections of the ſolids ABCD,

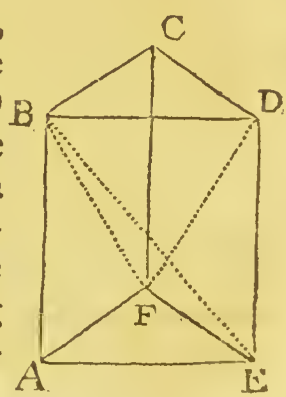


QRSTU, at all equal diſtances from the baſes, will [t] Lem. 5. be equal [t], that the ſolids themſelves will likewiſe be equal.

CASE II. If the perpendicular (DE) from the vertex to the plane of the baſe, falls beyond the limits of the triangle: Then, to the point E where it meets the plane, let BE and CE be drawn; and on BE let a triangle BF be deſcribed equal to ABC (or QRST), and let F, D be joined. So ſhall the pyramid CBFED, ſtanding on the baſe CBFE, be equal to the pyramid CABED, ſtand- [u] Lem. 4. ing on the equal baſe CABE [u]; from each of which, let the common pyramid CBED be taken away; and there will then remain the pyramid [w] Ax. 5. BFED = pyramid ABCD [w]: But the former of theſe is (*by caſe* 1.) equal to the cone QRSTU; therefore it is evident, that the latter ABCD will [x] Ax. 1. alſo be equal to the [x] cone QRSTU; and, conſequently, that all pyramids of the ſame altitude, ſtanding on equal triangular baſes, will be equal among themſelves [x]; ſeeing every ſuch pyramid is equal to a cone (QRSTU) of equal baſe and altitude.

THEOREM VII.

Every priſm (ABCDEFA) *having a triangular baſe* (AFE) *is equal to the triple of a pyramid of the ſame baſe and altitude.*



In the planes of the three ſides, let the diagonals BE, BF, FD be drawn. Then will the part FBCD of the priſm cut off by a plane extended by FB and FD, be a pyramid on the baſe BCD, having the ſame altitude with the [y] Def. 16. 7. priſm itſelf [y], both ſolids being contained between the ſame [z] pa- [z] Def. 6. 7. rallel planes AFE, BCD. More-

over,

over, the remaining part FABDE of the prifm, if a plane be extended by FB and FE, will be divided into the two pyramids FBAE, FDBE, which are equal to each other [a], as ftanding on the equal [b] triangular bafes ABE, BDE. But the former of thefe pyramids FBAE, if B be now confidered as the vertex thereof, will appear, alfo, to be equal to the firft mentioned pyramid FBCD [a], the two bafes AFE, BCD (as well as the altitudes) being equal [b]. Therefore, fince the three triangular pyramids (FBCD, FABE, FBDE) into which the prifm is refolved, are all equal to each other; the propofition is manifeft.

[a] 6. 8.
[b] 24. 1.

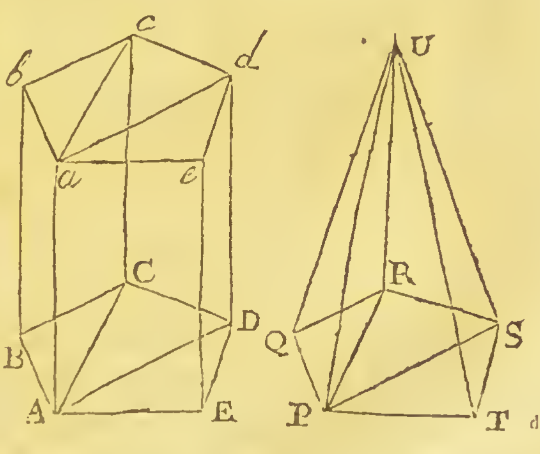
COROLLARY.

Hence, every prifm having a triangular bafe, is equal to the triple of any pyramid of the fame altitude, ftanding upon an equal triangular bafe [c].

[c] 6. 8. and Ax. 1.

THEOREM VIII.

*If a prifm (*AbcE*) and a pyramid (*PQRSTU*) ftand upon equal and fimilar bafes (*ABCDE, PQRST*), and have both the fame altitude; the prifm will be equal to the triple of the pyramid.*



If the bafes be refolved into triangles, ABC, ACD &c. it is manifeft, that BbacCA will be a prifm, on the bafe ABC; becaufe Cc being equal and parallel to [d] Aa, AacC will be a parallelogram [e] (as well as BbaA and BbcC [f]). There-

[d] Def. 6. and 8. 7.
[e] 26. 1.
[f] Def. 6. 7.

Therefore $BbacCA$ is equal to the triple of the
g Cor. to 7. 8. pyramid PQRU[g], ſtanding on an equal[h] baſe
h Hyp. and Ax. 10. (PQR). And, in the ſame manner, the priſm $AacdDC$, on the baſe ACD, is equal to the triple of the pyramid PRSU, on the equal baſe PRS; and ſo on. Therefore, alſo, ſhall the whole priſm Ad, on the baſe ABCDE, be equal to the triple of the whole pyramid PQRSTU, on the equal baſe PQRST.

COROLLARY I.

Hence, all pyramids having the ſame baſe and altitude, are equal; being like parts of one and the ſame priſm.

COROLLARY II.

Hence, alſo, all priſms having the ſame baſe and altitude, are equal; being equimultiples of one and the ſame pyramid.

COROLLARY III.

Therefore it appears, that every priſm inclining on its baſe, as well as every upright one, is equal to a rectangular parallelepipedon of equal baſe
k 20. 7. and Ax. 1. and altitude[k]; and, conſequently, that all priſms whatever, having equal baſes, and altitudes, are equal to each other[k]: which muſt be alſo true in pyramids and cones, every ſuch ſolid being ſubtriple to a priſm, or cylinder, of the ſame baſe and
l 8. 8. and 6. 8 altitude[l].

COROLLARY IV.

Hence it alſo follows, that whatever is demonſtrated in the 21ſt, 22d, and 23d Theorems of the preceding Book, concerning the proportion of priſms, holds equally in pyramids and cones; *theſe* being like parts of *thoſe*[l].

COROL.

COROLLARY. V.

It follows, moreover, that all correſponding fruſtums of pyramids and cones of the ſame altitude, are alſo, in proportion, as their baſes. For, the ſection[l], at all equal altitudes, being in that proportion[m], the parts cut off (as well the wholes) will be in the ſame proportion[n]; and, conſequently, the remaining parts likewiſe[o].

[m] Lem. 5. [n] Cor. 4. [o] 3. 4.

COROLLARY VI.

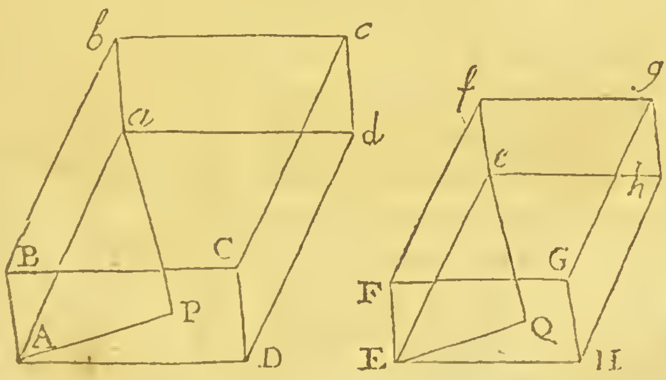
Laſtly, it will appear, that all cones, which have their altitudes and the diameters of their baſes directly proportional, are in the triplicate ratio of their altitudes[p]; being to each other in the ſame proportion with priſms of equal baſe and altitude, whereof they are like parts[q].

[p] 24. 7. [q] Cor. 3. and Cor. to 1. 4.

THEOREM IX.

All ſimilar priſms, and pyramids, are in the triplicate ratio of their altitudes.

From the extremes of the homologous ſides A*a*, E*e*, upon the baſes ABCD, EFGH of the propoſed ſolids A*c*, E*g*, let fall the perpendiculars



*a*P, *e*Q. The angle BAD being = FEH, BA*a* = FE*e*, and DA*a* = HE*e*[p], thence is *a*P : *e*Q : : *a*A : *e*E[q] : : AB : EF[r] : : AD : EH[r]. Therefore two

[p] Def. 5. 7. [q] 14. 7. [r] Def. 14. 4.

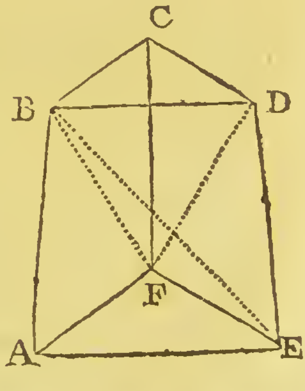
two upright prifms conftituted on the bafes ABCD, EFGH, of the fame altitudes (*a*P, *e*Q) with the two folids A*c*, E*g*, will be fimilar, the one to the
r Def. 5. 7. other [r]; and, therefore, in the triplicate ratio of the
s 24. 7. altitudes [s]. But the folids A*c*, E*g*, when taken as prifms, are refpectively equal to the faid upright ones; and, when taken as pyramids, are like parts
t Cor 3. to 8. 8. of them [t]. Therefore the folids A*c*, E*g* are alfo in the triplicate ratio of the altitudes *a*P and *e*Q [u].
u Cor. 1. 4.

COROLLARY.

Becaufe *a*P : *e*Q : : A*a* : E*e* : : AB : EF &c. it follows, that all fimilar prifms, and pyramids, are to one another, in the triplicate ratio of the homologous fides of the like planes by which they
y Cor. 1. to 5. 4. are bounded [y].

THEOREM X.

The fruftum (ABCDEFA) *of any pyramid having a triangular bafe, is equal to a whole pyramid, of the fame bafe and altitude, together with two other pyramids that are, in proportion thereto; the one, as any fide* (BD) *of the upper bafe* (BCD) *is to its correfpondent* (AF) *of the lower bafe* (AFE); *and the other, as the fquare of the former fide is to the fquare of the latter.*



In the planes of the three fides, let the diagonals BE, BF, FD be drawn. Then will the part FBCD of the fruftum, cut off by a plane extended by FB and FD, be a pyramid, on the bafe BCD, having the fame altitude with the fruftum
z Def. 16. 7. itfelf [z], both folids being contained between the fame [a] pa-
a Def. 15. 7. rallel planes AFE, BCD.

More-

Moreover, the remaining part FABDE of the frustum, if a plane be extended by FB and FE, will be divided into the two pyramids FBDE, FABE, having the same ratio, one to the other, as their bases BED, ABE[b], or as BD to AE[c]. But the latter of these pyramids (BAFE), taking B as the vertex thereof, has the same base and altitude with the frustum given: And the pyramid FBCD[d] (first mentioned) is therefore, in proportion thereto, as the base BCD to the base AFE[d], that is (because the bases are similar[e]), as BD^2 to[f] AE^2: *whence the proposition is manifest.*

[b] 21. 7. and Cor. 4. to 8. 8.
[c] 7. 4.
[d] 21. 7. and Cor. 4. to 8. 8.
[e] Cor. to Lem. 5.
[f] 26. 4.

COROLLARY.

Since, of the three solids (FABE, FBDE, FBCD) into which the proposed frustum is divided, the ratio of the first and third, is the duplicate of that of AE to BD, or of the ratio of the first to the second[g]; it is evident, that these three solids are proportionals[h] From whence it appears, that the frustum of any triangular pyramid is equal to two (whole) pyramids of the same altitude, on bases equal to the two opposite bases of the frustum, and to a third pyramid, which is a mean proportional between the two former. And it is also evident, that whatever is above demonstrated, in relation to triangular pyramids, must hold equally in all pyramids and cones, whatever: Because every such solid is equal to a triangular pyramid, of equal base and altitude[i]; and every frustum of the one, also equal to the corresponding frustum of the other[k].

[g] Cor. to 27. 4.
[h] Def. 7. 4.
[i] Cor. 3. to 8. 8.
[k] Cor. 5. to 8. 8.

LEMMA 6.

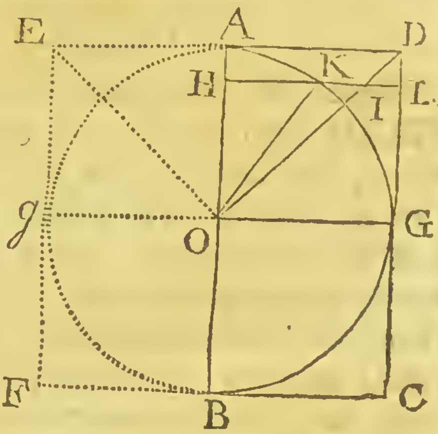
If with radii, respectively equal to the three sides of any right-angled triangle, three circles be described; that whose radius is equal to the hypothenuse, will be equal to both the other two, taken together.

It

It has been proved (*in Theor.* 3.) that circles are in proportion, as the ſquares of their radii; therefore the demonſtration here, is the very ſame, as in ſimilar right-lined figures (*Theor.* 29. *Book* 4.): Which (if neceſſary) you may reconſult.

THEOREM XI.

Every ſphere is two-thirds of its circumſcribing cylinder (Or, of a cylinder of equal diameter and altitude.)



Let AB be the axis about which the ſphere and cylinder are generated, by the revolution of the ſemi-circle AGB and the rectangle
[l] Def. 12. and 14. ADCB[l]; let HL be any right line perpendicular to AB, meeting DC in L, and the periphery of the ſemicircle in K; and from the center O, let OK and OD; interſecting HL in I, be drawn.

[m] Hyp. [n] 4. 1. [o] 14. 4. Since AO is = AD[m], and HI parallel to AD[n], therefore is HI = OH[o]: But OHK being right-angled at H, the circle whoſe radius is OH (or HI) will (*by the preceding Lem. and Ax.* 5.) be equal to the difference of the two circles whoſe radii are OK
[p] Lem. 6. (HL) and HK: Or, in other words, the circle[p] deſcribed by HI, or the ſection of the cone generated by the triangle AOD, in its revolution about the axis AB, will be equal to the difference of the two circles generated by HL and HK; that is,
[q] Ax. 2. 1. equal to the *annulus* deſcribed by KL[q], or the ſection of the ſolid which remains, when the ſphere is taken out of the cylinder. Therefore, ſeeing theſe

these two sections are, every-where, equal to each other, the solids themselves will likewise be equal [r]; that is, the cone (EOD) will be equal to the excess of the cylinder (GDE*g*) above the inscribed hemisphere (GA*g*): whence, as the cone, or excess, is one third part of the cylinder [s], the hemisphere must necessarily be equal to the two remaining thirds. And what is here proved, with respect to the halves of the proposed solids, holds equally in the wholes. Therefore every sphere is two-thirds of its circumscribing cylinder.

[r] Lem. 4.

[s] 6. 8. and Cor. 3. to 8. 8.

COROLLARY I.

Hence, a cone, hemisphere and cylinder, of the same altitude, and standing upon equal bases, are in proportion, as the numbers 1, 2 and 3, respectively.

COROLLARY II.

Hence it also appears, that all spheres are to each other in the triplicate ratio of their diameters [t]; being in the same proportion as the circumscribing cylinders, whereof they are like parts.

[t] Cor. to 5. 8. and 24. 7.

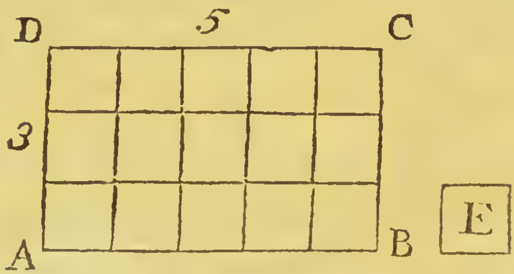
END *of the* ELEMENTS.

OF THE

MENSURATION

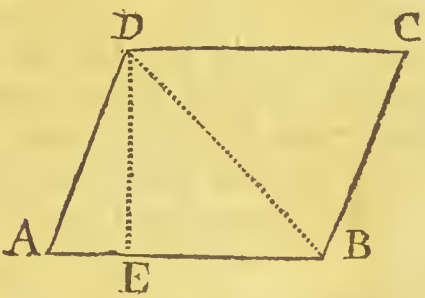
OF

Superficies and Solids.

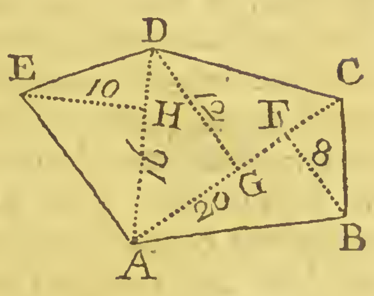


EVERY quantity is meafured by fome other quantity of the fame kind; as a line by a line, a furface by a furface, and a folid by a folid: And the number which fhews how often the leffer, called *the meafuring unit*, is contained in the greater, or quantity meafured, is called *the content of the quantity fo meafured.* Thus, if the quantity to be meafured be the rectangle ABCD, and the little fquare E, whofe fide is one inch, be the meafuring unit propounded; then, as often as the faid little fquare is contained in the rectangle, fo many fquare inches the rectangle is faid to contain: So that, if the length DC be fuppofed 5 inches, and the breadth AD 3 inches; the content of the rectangle

rectangle will be 3 times 5, or 15 ſquare inches: Becauſe, if lines be drawn parallel to the ſides, at an inch diſtance one from another, they will divide the whole rectangle ABCD into 3 times 5, or 15 equal parts, of one inch each. And, generally, whatever the meaſures of the two ſides may be, it is evident *(from El. 7. of* 4.) that the rectangle will contain the ſquare E, as many times as the baſe AB contains the baſe of the ſquare, repeated as often as the altitude AD contains the altitude of the ſquare. Therefore, *to find the content of any rectangle, multiply the baſe by the altitude, and the product will be the anſwer.* Thus, let the length be 18 inches, and the breadth 15; then the content will be 15 times 18, or 270 ſquare inches.



The method of finding the content of a rectangle being thus known, the content of any parallelogram ABCD, or triangle ABD, will alſo be known; the former of theſe figures being equal to a rectangle of the ſame baſe and altitude; and the latter equal to the half of ſuch a rectangle (by Cor. 2. to 2, 2.). Therefore, *multiply the baſe by the perpendicular, for the content of any parallelogram; and the baſe by half the perpendicular, for that of any triangle.* Thus, for example, let the baſe AB be 18 feet, and the perpendicular DE 12 feet; then the content of the parallelogram will be 216, and that of the triangle 108, ſquare feet.



From the manner of finding the area of a triangle, the area of any right-lined plain figure, as ABCDE, may be determined, by dividing the whole into triangles, and finding the content of each triangle. Thus, let the dividing lines AC and AD, be 20 and 16 inches, and the perpendiculars BF, DG, EH, falling thereon, 8, 12, and 10, reſpectively; then, the content of the

$$\text{Triangle}\left\{\begin{array}{c}\text{ABC}\\ \text{ACD}\\ \text{ADE}\end{array}\right\}\text{being}\left\{\begin{array}{c}80\\ 120\\ 80\end{array}\right\}\text{, it is evi-}$$

dent, that the content of the whole figure will be the ſum of all theſe, or 280 ſquare inches. But, when the given lines are expreſſed by fractions, or very large numbers, the work will be ſomewhat ſhortened, by finding the content of every two triangles, having the ſame baſe, at one operation; that is, by firſt adding the two perpendiculars together, and then multiplying half their ſum by the common baſe of the two triangles. Thus, in the laſt example, the half-ſum of the two perpendiculars BF and DG being 10, if this number be, therefore, multiplied by 20 the meaſure of the common baſe AC, the product, which is 200, will be the content of the trapezium ABCDA; to which 80, the content of the triangle ADE, being added; the ſum will be 280, the ſame as before. But, if the polygon propoſed be a regular one, that is, one whoſe ſides, and angles are all equal, the ſhorteſt way

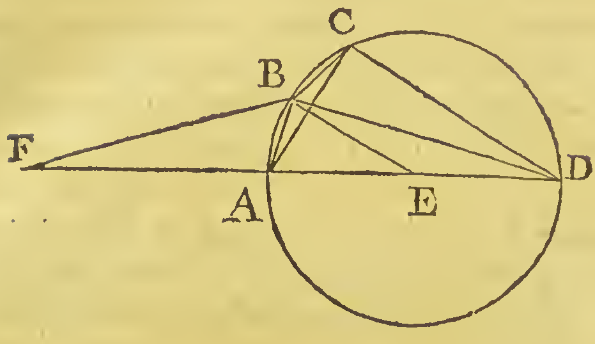
way of all, is, to *multiply half the ſum of all the ſides by the length of the line drawn from the middle of any ſide to the center of the polygon.* The reaſon of which is obvious, from the demonſtration to *Theor.* II. *B.* VIII.

Having ſhewn how the area of any right-lined figure may be computed, it will be proper here, to ſay ſomething with regard to the area, and periphery of the circle.

It is well known, that to determine the *true* area of a circle, and to find a right-line *exactly* equal to the circumference thereof, are looked upon, by mathematicians, as abſolutely impoſſible: But, though neither the one nor the other can be *accurately* known, yet ſeveral Ways have been invented by which they may be approximated, to any aſſigned degree of exactneſs. That, which I am now going to lay down, though leſs expeditious than ſome others, ſeems, nevertheleſs, to be the moſt proper for this place, as depending on the moſt ſimple and evident principles: I ſhall therefore begin with premiſing the following

LEMMA.

If AD *be a diameter, and* AB, BC *two equal arcs of the ſame circle, and if the chords* DB, DC *be drawn; then, I ſay, that* $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$.



For, if in DA produced, there be taken AF = DC, and BF, BA, BC and the radius BE be drawn; then, the external angle FAB, of the trapezium ABCD, being equal to the internal oppofite angle DCB (*by* 17. 3.) alfo AF = DC, and AB = CB (*by Hyp.*); it is evident, that FB is alfo = DB, and confequently the angle F = FDB = DBE: And fo the ifofceles triangles DEB, DBF being equiangular, it will be as DE ($\frac{1}{2}$AD) : DB : : DB : DF (DC + AD); and confequently $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$. *Q. E. D.*

COROLLARY.

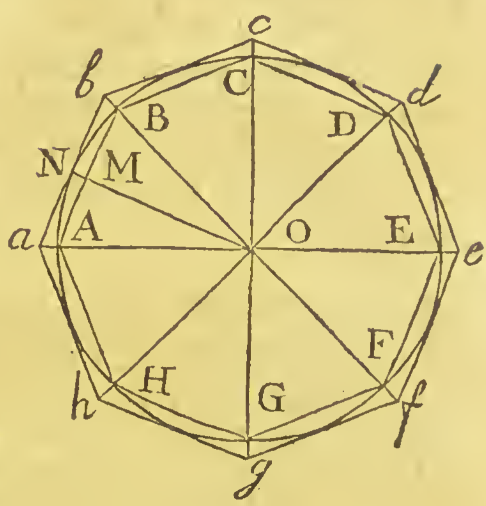
Hence, if the diameter AD be denoted by the number 2, the chord DB will be denoted by $\sqrt{DC + 2}$: whence, it appears, that, *if the meafure of the fupplemental-chord of any arch be increafed by the number* 2, *the fquare-root of the fum will be the fupplemental chord of half that arch.*

Now, to apply this to the matter propofed, that is, to the finding of the area and circumference of the circle; let the arch ABC be taken equal to $\frac{1}{3}$ of the femi-periphery ACD; then will the chord AC be equal to the radius AE (*by* 29. 5.); and

and therefore, ſince ACD is a right angle (*by* 13. 3.) DC^2 ($= AD^2 - AC^2$, *by* 8. 2.) will be $= 4 - 1 = 3$; and conſequently $DC = \sqrt{3} = 1{,}7320508075$, *&c.* Wherefore, ſeeing the ſupplemental chord of $\frac{1}{3}$ of the ſemi-periphery is 1,7320508075, we ſhall, by the preceding Corollary,

$$\text{have}\left\{\begin{array}{l} \sqrt{2+1{,}7320508075}=1{,}9318516525 \\ \sqrt{2+1{,}9318516525}=1{,}9828897227 \\ \sqrt{2+1{,}9828897227}=1{,}9957178465 \\ \sqrt{2+1{,}9957178465}=1{,}9989291743 \\ \sqrt{2+1{,}9989291743}=1{,}9997322757 \\ \sqrt{2+1{,}9997322757}=1{,}9999330678 \\ \sqrt{2+1{,}9999330678}=\sqrt{3{,}9999330678} \end{array}\right\} \text{for the ſupplemental chord of} \left\{\begin{array}{c} \frac{1}{6} \\ \frac{1}{12} \\ \frac{1}{24} \\ \frac{1}{48} \\ \frac{1}{96} \\ \frac{1}{192} \\ \frac{1}{384} \end{array}\right\} \text{of the ſemi-periphery.}$$

Now, therefore, ſince it is found that 3,9999330678 is the ſquare of the ſupplemental-chord of $\frac{1}{384}$ of the ſemi-periphery, let this number be ſubſtracted from 4 the ſquare of the diameter, and the remainder 0,0000669322 will be the ſquare of the chord of the ſame arch; therefore the chord itſelf being $= \sqrt{0{,}0000669322} = 0{,}0081812$1, let this number be multiplied by 768, or twice 384, and the product 6,28317 will be the perimeter of a regular polygon of 768 ſides, inſcribed in the circle; which, as the ſides of the polygon very nearly coincide with the circumference of the circle, muſt alſo expreſs the length of the circumference itſelf, very nearly.



But, in order to fhew how near this is to the truth, let AB reprefent one fide of a regular polygon of 768 fides, infcribed in the circle (whofe length, we have found above, to be 0,00818121) and let *ab* be a fide of another fimilar polygon, defcribed about the circle; and from the center O let ON be drawn, bifecting AB and *ab* in M and N; Then, fince AM is $= \frac{1}{2}$AB $=$ 0,0040906, and AO $= 1$, it is plain that OM^2 ($AO^2 - AM^2$) will be $=$ 0,99998327, and confequently OM $=$ 0,99999163; whence, becaufe of the fimilar triangles AOB, *aOb*, &c. we have 0,99999163 (OM) : 1 (ON) : : AB : *ab* : : 6,28317 (the perimeter of the infcribed polygon) : 6,28322 the perimeter of the circumfcribed polygon. But the circumference of the circle being greater than the perimeter of the infcribed polygon, and lefs than that of the circumfcribed one, it muft, confequently, be greater than 6,28317, and lefs than 6,28322; and muft, therefore, be equal to 6,2832, very near; fince this number exceeds the perimeter of the infcribed polygon by no more than 0,00003, and is lefs than the perimeter of the circumfcribed one by 0,00002, only.

From the periphery thus found, the area of the circle will alfo be known; being equal to the product of half the periphery into the radius (*by* 2. 8.) that is, $= 3{,}1416 \times 1 = 3{,}1416$.

There-

Therefore, since it is proved (*in Theor.* 3 *and* 4. *of* 8.) that the peripheries of circles are in proportion as their diameters, and the circles themselves as the squares of those diameters; it follows, that, as 2 is to 6,2832, or as 1 to 3,1416 : : the diameter of any circle to its periphery; and as 4 to 3,1416, or as 1 to 0,7854 : : the square of the diameter to the area.

But, if you had rather have the proportions in whole numbers, and the case proposed does not require any great degree of accuracy; then, instead of the foregoing, those of *Archimedes* may be used, *viz.* 7 : 22 : : diam. : circumf. and 14 : 11 : : square diam. : area. Which proportions differ but little from those above, as will appear from the following example: wherein the diameter of a circle being given 28, its circumference and area are required. Here, according to the first proportions, I multiply 28 by 3,1416 for the circumference, and the square of 28 (or 784) by 0,7854 for the area; and there results 87,964 and 615,75, respectively. But, according to the proportions of *Archimedes*, the circumference will be found equal to 88, and the area 616; which differ very little from the former.

By knowing the proportion between the diameter of a circle and the circumference, and between the square of the diameter and the area, the convex superficies of solid bodies may be determined. Thus,

The convex superficies of a cylinder is found, *by first finding the circumference of the base, and then multiplying by the altitude of the solid.* Therefore, if to that product, the area of the two circular ends be added, the sum will be the whole superficies of the cylinder.

To find the convex ſuperficies of a cone, *multiply half the length of the ſlant ſide thereof by the circumference of the baſe.*

The convex ſuperficies of any fruſtum of this ſolid is found, *by multiplying the ſum of the peripheries of the two ends into half the length of the ſlant ſide of the fruſtum.*

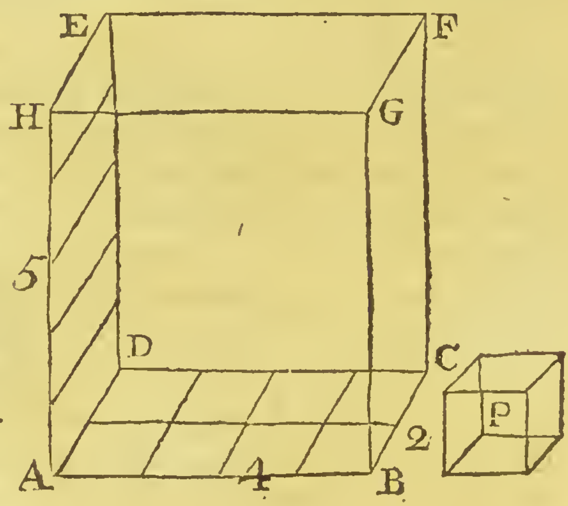
To find the ſuperficies of a ſphere, *multiply the periphery of the greateſt, or generating, circle by its diameter :* Or, *multiply the ſquare of the diameter by* 3,1416.

The convex ſuperficies of any ſegment of a ſphere is found, *by multiplying the periphery of the greateſt circle of the ſphere into the altitude of the ſegment.*

The demonſtration of theſe laſt rules, for finding the curve ſurfaces of ſolid bodies (which is not given in the *Elements*, for reaſons mentioned hereafter) is inſerted at the end of this ſection.

Of the Mensuration of Solids.

As every ſuperficies is meaſured by a ſquare, whoſe ſide is unity (as one inch, one foot, one yard, *&c*) ſo every ſolid is meaſured by a cube whereof the ſide is alſo an unit. Thus,



let

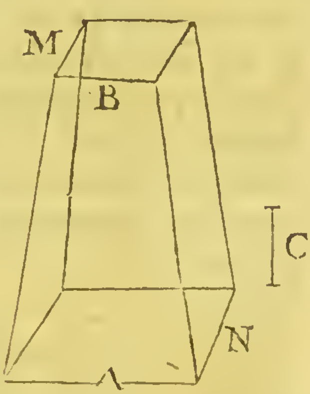
let the folid to be meafured, be the rectangular parallelepipedon AF, and let the cube P, whofe fide is one inch, be the *meafuring unit*; alfo let the length AB, of the bafe AC, be 4 inches, the breadth BC 2 inches, and the altitude AH of the folid 5 inches: Then, becaufe the area of the bafe ABCD is 2 times 4 (or 8) fquare inches, it is eafy to conceive, that, if the folid were to be only one inch high (inftead of 5), the content thereof would be juft the fame number (8) of cubical inches; becaufe then, upon the eight equal fquares into which the whole bafe ABCD is divifible, a cube of one inch might be erected, fo as to compofe a parallelepipedon on that bafe, of one inch high. Therefore, feeing that the content of the folid, at one inch high, is 8 cubical inches, the whole content at 5 inches high, muft confequently be 5 times 8, or 40 cubical inches (fince the whole folid AF may be confidered, as compofed of 5 fuch heights of cubes, one ranged above another.) And, generally, whatever the dimenfions may be, it is manifeft (*from* 21 and 22. of 7.) that the parallelepipedon will contain the cube P, as many times as the bafe ABCD contains the bafe of the cube, repeated as often as the altitude AH contains the altitude of the cube. Therefore *the content of any parallelepipedon will be found, by multiplying the area of the bafe by the altitude of the parallelepipedon.* Thus, for example, if the two dimenfions of the bafe be 16 and 12 inches, and the height of the folid 10 inches; then, the area of the bafe being 192, the content of the folid will be 1920 cubical inches.

From the content of a parallelepipedon, thus known, *that* of a prifm, or a cylinder, will likewife be known; every fuch folid being (*by* 20. 7. *or* 5. 8.) equal to a parallelepipedon of equal bafe, and

and altitude. Therefore, *multiply the area of the baſe* (found by the rules for ſuperficies) *into the height of the priſm, or cylinder, and the product will be the content.*

Hence the content of any pyramid, or cone, is alſo obtained; being (*by Cor.* 3. *to* 8. 8.) equal to $\frac{1}{3}$ part of a priſm, or cylinder, of the ſame baſe and altitude. Therefore, *multiply the area of the baſe by* $\frac{1}{3}$ *of the altitude, and the product will be the anſwer.*

Every ſphere being (*by* 11. 8.) equal to $\frac{2}{3}$ parts of a cylinder of the ſame diameter and altitude; *the content of any ſphere will, therefore, be found, by multiplying the area of its greateſt, or generating, circle into* $\frac{2}{3}$ *of its diameter:* Or (becauſe the area of ſuch circle is to the ſquare of the diameter, in proportion as 0,7854 to 1), *let the cube of the diameter be multiplied by the fraction* ,5236 (= $\frac{2}{3}$ of 0,7854), *and the product will be the content.* Thus, if the meaſure of the diameter be 20, the cube thereof will be 8000; which, multiplied by ,5236, will give 4188,8 for the meaſure of the ſphere's ſolidity.

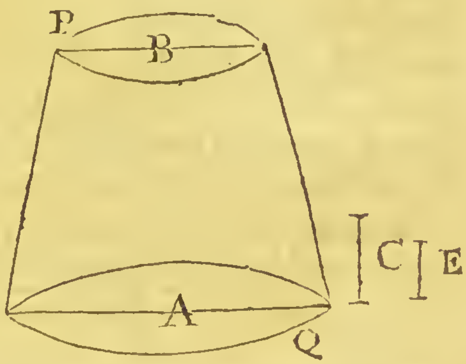


The manner of finding the content of any fruſtums of the ſolids above determined, is collected from *Theor.* 10. *and* 11. *B.* VIII. Let the fruſtum (MN), firſt propoſed, be that of a pyramid; then, having found the content of a whole pyramid, of the ſame given baſe and altitude; ſay, as any ſide A of the lower end or baſe, is to its correſpondent B of the upper, ſo is the ſaid con-

tent

content to a fourth-proportional; and, as A is (again) to B, ſo is the quantity laſt found to another proportional: which two proportionals, added to the content firſt determined, will give the true content of the fruſtum. But when the oppoſite baſes of the fruſtum are ſquares, the rule will be more ſimple, and put on a better form: For then the area of the baſe being A^2, the content of a whole pyramid thereon, of the ſame altitude with the fruſtum, will be equal to the parallelepipedon $C \times A^2$, C being $\frac{1}{3}$ of the given altitude of the fruſtum. But $A : B :: C \times A^2 : C \times A \times B$ (*by* 22. 7.) and $A : B :: C \times A \times B : C \times B^2$. Therefore $C \times A^2 + C \times A \times B + C \times B^2$ ($= C \times \overline{A^2 + A \times B + B^2}$, *by Schol. to* 20. 7.) is the true content, in this caſe.

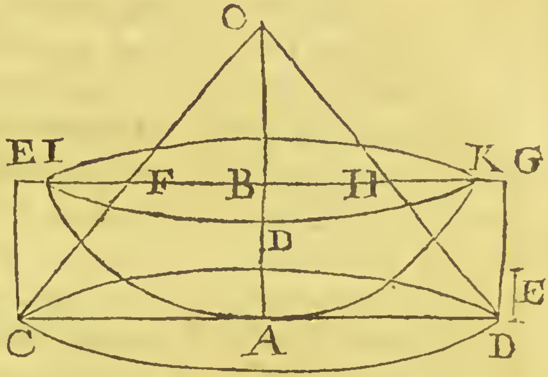
Hence, *to find the content of the fruſtum of any ſquare pyramid, add the product of the two ſides of the lower and upper ends to the ſum of their ſquares, and then multiply the aggregate by* $\frac{1}{3}$ *of the fruſtum's height.*



From the content here found, *that* of any conical fruſtum (PQ) is readily obtained; being in proportion to the content ($C \times \overline{A^2 + A \times B + B^2}$) of the fruſtum of a ſquare pyramid circumſcribing it, as the baſe of the former is to the baſe of the latter (*by Cor.* 5 *to* 8. 8.), or as the fraction ,7854 is to *unity*: And ſo, will be equal to the ,7854 part of $C \times \overline{A^2 + A \times B + B^2} = E \times \overline{A^2 + AB + B^2}$; by taking $E = ,7854 \times C =$ the ,2618 part of the whole given altitude. Therefore,

fore, *to find the content of any fruſtum of a cone, add the product of the diameters of the two ends to the ſum of their ſquares; then multiply the aggregate by the fruſtum's height, and the product, again, by the fraction* ,2618.

Hence, and from *Theor.* 11. *B.* VIII. a Rule for finding the content of any ſegment IAK of a ſphere, may alſo be deduced: For, it appears, *from thence*, that the ſegment propoſed, IAK, is equal to the difference between a conical fruſtum FCDH and a cylinder ECDG of the ſame altitude, ſtanding upon a baſe, whoſe radius CA is equal to that (AO) of the ſphere itſelf. But the content of the fruſtum FCDH, if the two diameters CD, FH be repreſented (*as above*) by A and B, and the ,2618 part of the altitude (D) by E, will be $= E \times \overline{A^2 + A \times B + B^2}$ (that is, equal to a parallelepipedon whoſe altitude is E, and baſe $= A^2 + A \times B + B^2$): And the content of the cylinder ECDG will be $= 3E \times A^2$, or $E \times 3A^2$. Therefore the difference (or the content of the ſegment IAK) will be $= E \times \overline{2A^2 - A \times B - B^2}$ (*Schol. to* 20. 7. *and Ax.* 5. 1.) But $2A^2 - A \times B - B^2$ is compoſed of $A^2 - A \times B$ and $A^2 - B^2$; whereof the former part $A^2 - A \times B$ is $= \overline{A - B} \times A$ (*by* 5. 2.) $= 2D \times A$ (becauſe $A - B$ (or $CD - FH$) $= FE + HG = 2\overline{AB}$ or $2D$); and the latter $A^2 - B^2 = \overline{A - B} \times \overline{A + B}$ (*by* 7. 2.) $2D \times \overline{2A - 2D}$: Whence the ſum of both will conſequently be $= 2D \times \overline{3A - 2D}$; and the content of the ſegment itſelf



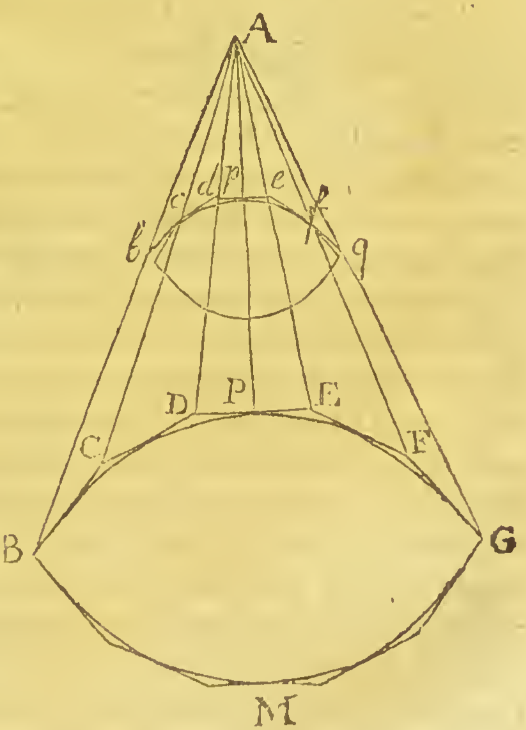
itself $= E \times 2D \times \overline{3A - 2D} = ,5236 \times D^2 \times \overline{3A - 2D}$ (becauſe $2E = ,5236D$).

Therefore, *to find the content of any ſegment of a ſphere, multiply the ſquare of the ſegment's height by the exceſs of thrice the ſphere's diameter above the double of that height; and then multiply by the fraction ,5236.*

The demonſtration of the rules for determining the ſuperficial content of the cylinder, cone and ſphere, and of their ſeveral ſegments, or fruſtums, is collected from the two *Lemmas* here ſubjoined.

LEMMA I.

The upper ſuperficies, or the area of all the ſides of a regular pyramid, in which a cone may be inſcribed, is equal to a rectangle under the perimeter of the baſe and half the length of the cone's ſlant ſide.



For, let BCDE, &c. be the baſe of the pyramid, and BPGM that of the inſcribed cone; and from the vertex A to the point P where any ſide DE of the polyg. touches the circle, let AP be drawn. Then, ſince the triangle ADE is $= \frac{1}{2}AP \times DE = \frac{1}{2}AB \times DE$; and as the like holds good with regard to every other ſide

of

of the pyramid, it is evident that the sum of all the sides, or the whole superficies of the pyramid (exclusive of the base) will be equal to $\frac{1}{2}AB \times \overline{DE + EF + \&c}$; that is, equal to a rectangle under $\frac{1}{2}AB$ and the whole perimeter of the base.

COROLLARY I.

Hence it will also appear, that all the sides of any frustum *Bg* of the pyramid, will be equal to a rectangle under half the length of each side and the sum of the perimeters of the two ends: For, the area of the side *DEed* being $= \frac{1}{2}Pp \times \overline{DE + de}$, or $\frac{1}{2}Bb \times \overline{DE + de}$ (*by* 4. 2.), the area of all the sides will, therefore, be $= \frac{1}{2}Bb \times \overline{DE + de + EF + ef + \&c.}$

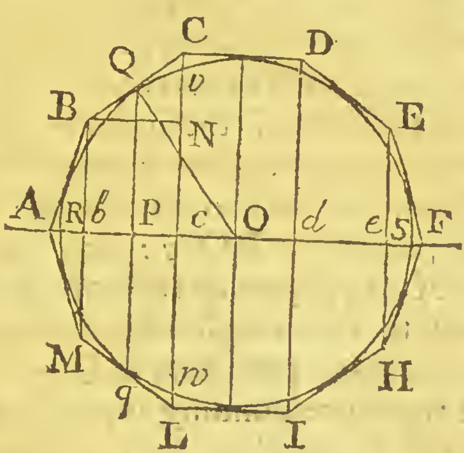
COROLLARY II.

Therefore, seeing that the foregoing conclusions hold universally, whatever the number of the sides may be; and as the pyramid, by increasing the number of its sides, approaches nearer and nearer, continually, to the inscribed cone, which is its limit; thence will the upper superficies of the cone (as well as that of the pyramid) be equal to a rectangle under half the length of its slant side and the perimeter of its base. And the convex superficies of any frustum of the cone will, *also*, be equal to a rectangle under half the length of its slant side and the sum of the peripheries of its two ends, or bases: Whence it likewise follows, that the convex surface of a cylinder will be equal to a rectangle under half its altitude and twice the periphery of its base (or under the whole altitude and once that periphery); because then the two ends are equal.—From this Corollary, the rules for finding the superficies of the cylinder and cone, are given.

LEM-

LEMMA 2.

If a regular polygon ABCDE, &c. *of an even number of sides, together with its inscribed circle* RQSq, *be supposed to revolve about the (produced) diameter* RS, *as an axis; the superficies of the solid generated by the polygon, will be equal to a rectangle under its axis* AF *and a right line equal to the circumference* RQSq *of the inscribed circle.*



From the center O, to the point of contact Q, of any side BC, let the radius OQ be drawn; also draw B*b*M, QP*q*, C*c*L, &c. perpendicular to AF, and BN perpendicular to CL.

Because the solid generated by the plane B*bc*C is the frustum of a cone, the convex superficies thereof, generated by BC, is equal to a rectangle under $\frac{1}{2}$ BC and the sum of the peripheries of the two circles described by B*b* and C*c* (*by Cor. 2, to the precedent*): But the sum of these two peripheries, as QP is an arithmetical mean between B*b* and C*c*, is equal to twice the periphery Q*q*; and therefore the convex superficies of the said frustum equal to $\frac{1}{2}$BC × 2 periph. Q*q* = BC × periph. Q*q*. But, because of the similar triangles OPQ, BNC, we have BC : BN (*bc*) : : OQ : PQ : : periph. RQS*q* : periph. Q*q* (*by* 4. 8); and consequently BC × periph. Q*q* = *bc* × periph. RQS*q* = the superficies

perficies generated by BC. By the very fame argument, the fuperficies generated by any other fide CD is $= cd \times$ periph. $RQSq$: Whence it is manifeft, that the fuperficies of the whole folid is $= \overline{Ab + bc + cd + \&c.} \times$ periph. $RQSq = AF \times$ periph. $RQSq$.

COROLLARY I.

Since the fuperficies of the folid is, univerfally, equal to $AF \times$ periph. $RQSq$, let the number of fides of the generating polygon be what it will; and as the faid fuperficies, by increafing the number of fides, approaches nearer and nearer, continually, to the fuperficies of the infcribed fphere, which is its limit; thence will the fuperficies of the fphere, itfelf, be alfo equal to a rectangle under its axis RS and periphery $RQSq$: and the convex fuperficies of any fegment thereof vRw, will likewife be equal to a rectangle under its axis (or height) Rc and the fame periphery $RQSq$; fince it is proved, that the correfponding fuperficies of CBAML, is univerfally equal to $Ac \times$ periph. $RQSq$.

COROLLARY II.

Hence it alfo appears, that the fuperficies of every fphere is equal to four times its generating circle: Becaufe (*by* 2. 8) the circle $RQSq = \frac{1}{2}RS \times \frac{1}{2}$periph. $RQSq = \frac{1}{4}RS \times$ periph. $RQSq$.

In deriving thefe conclufions, as well as thofe depending upon the preceding Lemma, the *Reader* muft have obferved, that fomething is affumed, which is not demonftrated in any part of thefe *Elements*. But this will not, I imagine, be confidered as a fault, by thofe who know, that it is impoffible to prove in a manner *perfectly* regular and geometrical, that a *curve furface*, of any kind, is equal

to

to a *plane-one* of an assigned magnitude. Plane surfaces are compared with one another, in virtue of the 10*th Axiom*; in which, whatever relates to the equality of plane figures, has its original. But no principles have been yet admitted into the *common*, or *lower Geometry*, whereby a curve-surface can be compared with a plane one; nor even by which the proportion of any one curve-line to a right-line can be known: Nor can it be demonstrated by all the Geometry in *Euclid's Elements*, that the periphery of a circle is less than the perimeter of its circumscribing square.—We can determine the proportion of solids bounded by curve-surfaces, by describing other solids in, and about them, so as to differ less from them, than by any assigned part however small. But in comparing of the surfaces, this method fails; because, let the number of sides of the inscribed, or circumscribed solid be ever so great, or let the solid itself approach ever so near to the proposed one; the two surfaces, after all, *will have no part in common* on which a demonstration can be formed, but will still be distinct things. Before such a comparison can possibly be made, in a regular and scientific manner, *new principles* must be laid down: But *these* belong to, and are best supplied in the *Modern Geometry*, or *Method of Fluxions*.

OF THE

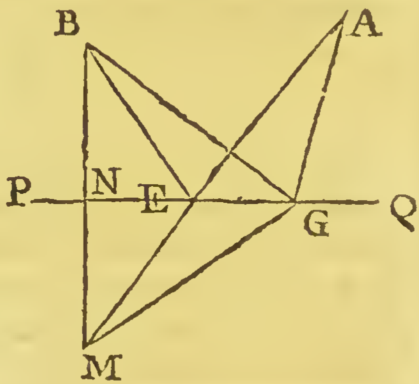
MAXIMA and MINIMA

OF

Geometrical Quantities.

THEOREM I.

If from two given points A, B, *on the ſame ſide of an indefinite line* PQ *(in the ſame plane with them) two lines* AE, BE *be drawn to meet on, and make equal angles* AEQ, BEP *with the ſaid line* PQ; *the lines ſo drawn, taken together, ſhall be leſs than any other two* AG, BG, *drawn from the ſame points to meet on the ſame line* PQ.



For, let BNM be perpendicular to PNQ, and let AE be produced to meet it in M, alſo let MG be drawn.

Then the triangles MNE, BNE, having the angle MEN (= AEQ[a]) = BEN[b], MNE = BNE[c], and NE common to both; have alſo MN = BN, and ME = BE

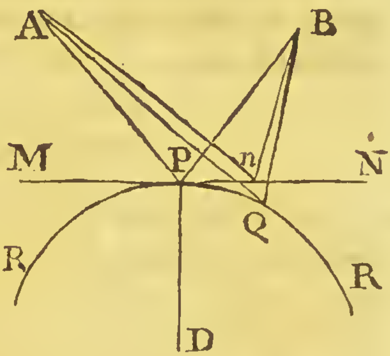
[a] 3. 1.
[b] Hyp.
[c] Conſtr.

BE[d]; whence alſo MG = BG[e]: But AM (AE + BE) is leſs than AG + MG[f], or, than its equal AG + BG. *Q. E. D.*

[d] 15. 1.
[e] Ax. 10. 1.
[f] 19. 1.

THEOREM II.

Of all right-lines AP, BP; AQ, BQ, *that can be drawn from two given points* A, B, *to meet, two by two, on the convexity of a given circle* RPQR; *thoſe two* AP, BP *taken together, ſhall be the leaſt, which make equal angles with the tangent* MPN (*or with the radius* DP) *at the point of concourſe* P.



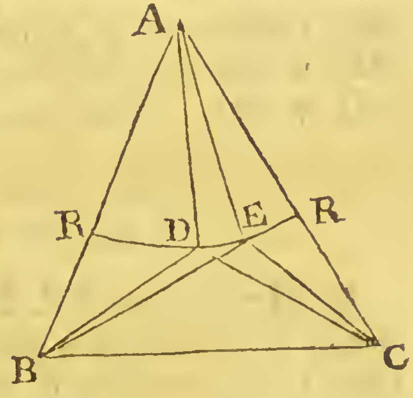
For, if to any point *n* in the part of the tangent intercepted by AQ and BQ, there be drawn A*n* and B*n*; then will AP + BP be leſs than *An* + B*n*[g], and A*n* + B*n* leſs than AQ + BQ[h]: Conſequently AP + BP is leſs than AQ + BQ. *Q. E. D.*

[g] Theor. 1.
[h] 23. 1.

This demonſtration holds equally true, when the curve RPR is ſuppoſed of any other kind; provided all tangents to it, fall intirely without the curve.

THEOREM III.

If, in a given triangle ABC, *a point is to be determined, ſo that the ſum of all the three lines drawn from thence to the three angles, ſhall be the leaſt poſſible; I ſay, the poſition of that point muſt be ſuch, that all the angles formed about it by thoſe lines, ſhall be equal among themſelves.*



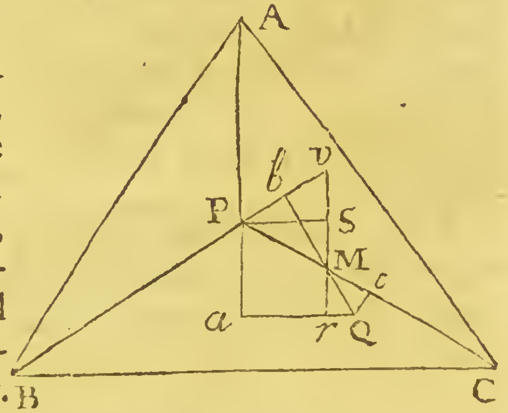
If you deny it, then let some point E, at which the angles BEA, CEA are unequal, be the required one.

Upon the center A, thro' E, let the circumference of a circle RER be described; and let D be that point in it, where the angles ADB and ADC are equal.

[i]Theor. 2. Because BD + CD is less than BE + CE[i], therefore is AD + BD + CD also less than AE + BE + CE[k]; *which is repugnant.* Therefore no point at which the angles are unequal, can be the required one. *Q. E. D.*

[k] Ax. 6. 1.

The same otherwise.



Let the point P be that, at which all the angles APB, APC, BPC are equal*; and from any other point Q, upon the lines forming them, let fall the three perpendiculars Q*a*, Q*b*, Q*c*. I say, first, that the sum of the three distances A*a*, B*b*, C*c*, intercepted by those perpendiculars, and the three given points A, B, C, will be equal to the sum of the three

* *The determination of the position of a point, at which, lines drawn from three given points, shall form any given angles, is given among the Geometrical Constructions, in the next section.*

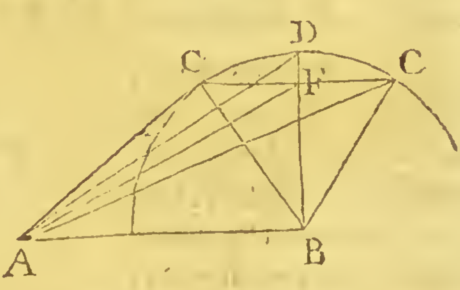
first

firſt diſtances AP + BP + CP. For, if thro' the interſection M of Pc and bQ, rv be drawn parallel to Aa, meeting Bb (produced) in v; and PS, parallel to aQ, in S; it is evident (becauſe the angle v = [l] BPa = vPM = $\frac{2}{3}$ of a right-angle [m]) that the triangle PvM is equilateral; and that the right-angled triangles QMc, QMr, having QMc (= PMb = vMb) = QMr, have alſo cM = rM; to which let MP = Mv be added; ſo ſhall cP = rv = aP + vS = aP + Pb. And, if to the firſt and laſt of theſe, AP + BP + Cc (be again) added; then will AP + BP + CP = Aa + Bb + Cc, *as was aſſerted:* Whence the Theorem itſelf is exceedingly obvious: For ſeeing that the ſum AP + BP + CP is but equal to the ſum of the three baſes Aa, Bb, Cc, it muſt neceſſarily be leſs than that of the three hypothenuſes AQ, BQ and CQ. *Q. E. D.*

[l] Cor. to 8. 1.
[m] Hyp.

THEOREM IV.

The greateſt triangle ABD *that can poſſibly be contained under two right-lines, given in length, and any other right-line joining their extremes, will be when the two given lines* AB, BD *make right angles with each other.*



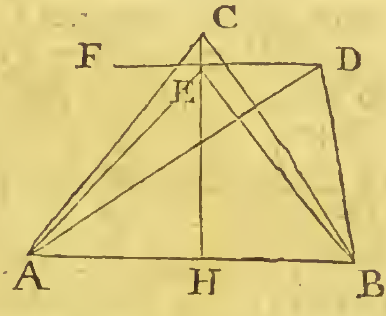
For, let BC be equal to BD, and the angle ABC either greater, or leſs than the right angle ABD; let alſo CF be drawn parallel to AB, meeting BD (produced, if neceſſary) in F, and let A, F, and A, C be joined.

Then the angle BFC being a right-one [n], it is evident that BC (BD) is greater than BF [o]; and therefore the triangle ABD, being greater than ABF [p], is alſo greater than its equal [q] ABC. *Q. E. D.*

[n] 5. 1.
[o] 20. 1.
[p] Ax. 2.
[q] Cor. to 2. 2.

THEOREM V.

Of all triangles ABC, ABD, *having the same base* AB, *and the sum of their other sides the same, the isosceles one* ACB, *is the greatest.*



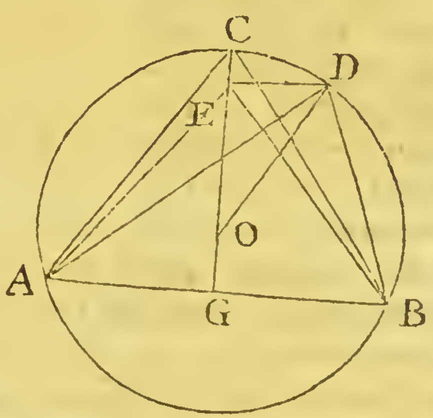
Let CH be perpendicular to AB, and DEF parallel to AB, interfecting HC (produced if need be) in E; likewise let AE and BE be drawn.

It is manifest that the angles AEF, BED are equal [r]; therefore AE + BE is less than AD + BD [s], or than its equal AC + BC [t]; and so the triangle AEB, falling within the triangle ACB [u], must be less than ACB [w]; and therefore ADB (= AEB [x]) must also be less than ACB. *Q. E. D.*

[r] 16. and 7. 1.
[s] Theor. 1.
[t] Hyp.
[u] 23. 1.
[w] Ax. 2 1.
[x] Cor. 1. to 2. 2.

THEOREM VI.

Of all triangles ABC, ABD *standing upon the same base* AB, *and having equal vertical angles* ACB, ADB, *the isosceles one* ACB *is the greatest.*



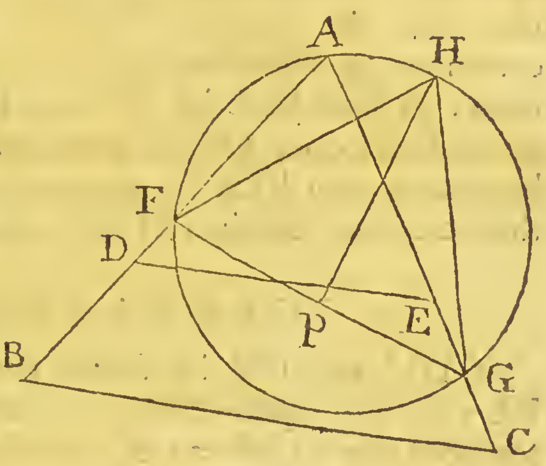
Let ACDB be a segment of a circle, in which the equal angles ACB, ADB are contained [y]; make CEG perpendicular, and DE parallel, to AB; from the center O draw OD, and let A, E and B, E be joined. It is evident that CG, not only bisects AB [z], but also passes through the center O [a]. Therefore, OD (OC) being

[y] 22. 5. and 11. 1.
[z] 16. 1.
[a] Cor. to 2. 3.

ing greater than OE [a], the triangle ACB will alſo be greater than AEB, or than its equal ADB [b]. *Q. E. D.* [a] 20 1. [b] Cor. 2. 2. and Ax. 2.

THEOREM VII.

Of all right-lines DE, FG *that can be drawn to cut off equal areas* ADE, AFG *from a given triangle* ABC, *that* DE *is the leaſt, which makes the triangle* ADE, *cut off, an iſoſceles one.*

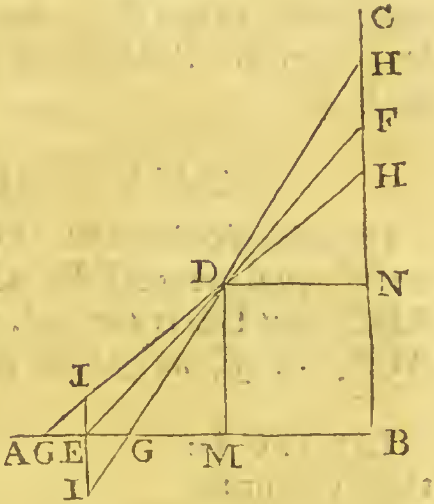


Let AFG be the circumference of a circle paſſing through the three points A, F, G; alſo let PH be perpendicular to FG, at the middle point P, meeting the circumference in H, and let FH and GH be drawn. The triangle FHG, being iſoſceles [c], is therefore greater than FAG [d], or than its equal [e] ADE: Whence, as the triangles FHG, ADE are equiangular [f], the baſe FG of the greater, muſt conſequently exceed the baſe DE of the leſſer. *Q. E. D.*

[c] Ax. 10. [d] Theor. 6. [e] Hyp. [f] Hyp. and Cor. to 19. 1.

THEOREM VIII.

Of all right-lines EF, GH, GH *that can be drawn thro' a given point* D, *between two right-lines* BA, BC *given in poſition; that* EF *which is biſected by the given point* D, *forms with them the leaſt triangle* (EBF).



For, if EI, parallel to BC, be drawn, meeting GH in I; the [h] equiangular triangles DFH and DEI, having DF $=$ DE [i], will be equal [k]; and DFH will therefore be leſs, or greater than DEG [l], according as BG is greater or leſs than BG. In the former caſe let DEBH common, be added to both; ſo ſhall FEB be leſs than HGB [k]. And if, in the latter caſe, DGBF be added, then will HGB be greater than FEB [m]; and conſequently FEB (in this caſe alſo) leſs than HGB. *Q. E. D.*

h 3. and 7. 1.
i Hyp.
k 15. 1.
l Ax. 2.
m Ax. 6.

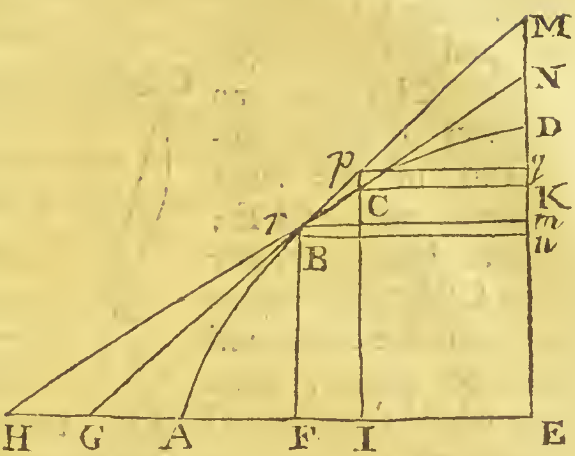
COROLLARY.

If DM and DN be drawn parallel to BC and BA; the two equal [o] triangles DEM, DFN, taken together (ſince EM $=$ DN [o] $=$ MB [p]) will be equal to the parallelogram DMBN [q]; and conſequently the parallelogram DMBN $= \frac{1}{2}$ BEF $\sqsupset \frac{1}{2}$ BGH. Whence it is manifeſt, that a parallelogram is always leſs than half a triangle in which it is inſcribed, except when the baſe of the one is half the baſe of the other; in which caſe the parallelogram is juſt half the triangle.

o 15. 1.
p 24. 1.
q Cor. to 2. 2.

SCHO-

SCHOLIUM.

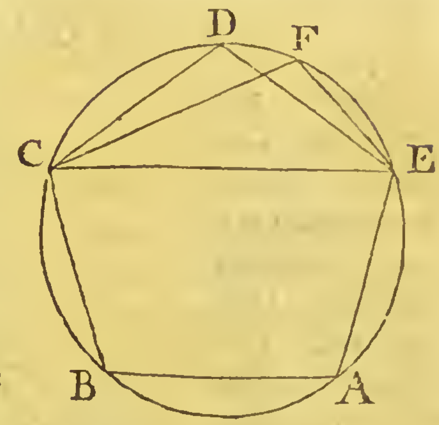


From the preceding Corollary alone, it may be very easily made to appear, that the least triangle EGM which can possibly be described about, and the greatest parallelogram EFBn that can be described in, any curve ABCD, concave to its axis AE, will be when the sub-tangent FG is equal to half the base EG of the triangle, or to the whole base EF of the parallelogram; and that the two figures will be in the ratio of *two* to *one*. For let HN be a side of any other circumscribing triangle (EHN) touching the curve in C, and meeting FBr in r: Then, the curve being concave to its axis, the point r will fall above B; whence, if rm be drawn parallel to Bn, then will EGM = 2BE ⊐ 2rE ⊐ EHN. Again, if IC, parallel to EM, be produced to meet GM in p, and CK and pq be drawn parallel to AE; then, also, will BE = ½EGM ⊏ pE ⊏ CE, *as was to be shewn.*

THEOREM IX.

Of all right lined figures, contained under the same number of sides, and inscribed in the same circle, that is the greatest whose sides are all equal.

For,



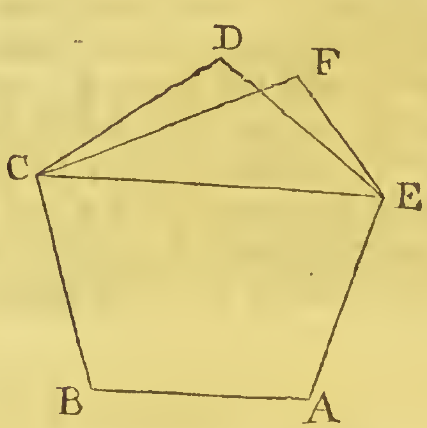
For, *if possible*, let some polygon ABCFE, whose sides CF, EF are unequal, be the greatest.

Let CDE be an isosceles triangle described in the same segment with CFE; which being greater than CFE [r], the whole polygon ABCDE will also be greater than the whole polygon ABCFE; *which is repugnant*. Therefore the polygon is the greatest when the sides are all equal.

[r] Theor. 6. and 11. 3

THEOREM X.

Of all right-lined figures, contained under the same perimeter, and number of sides, the greatest is, when the sides are all equal.

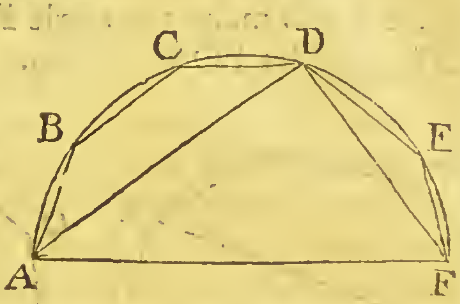


For, if ABCDE be the greatest possible, the triangle CDE must, manifestly, be greater than any other triangle CFE upon the same base, whereof the sum of the other sides is also the same. But, *by Theorem* V. the greatest triangle, when the base and the sum of the sides are given, is that whose sides are equal: Therefore DC and ED are equal. In the same manner it appears that BC = CD, *&c.* *Q. E. D.*

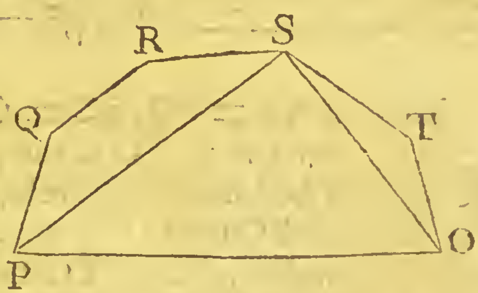
THEO-

THEOREM XI.

If all the ſides of a polygon, except one, be given in length, and their poſition be required, ſo as to make the polygon itſelf the greateſt poſſible; I ſay, their poſition muſt be ſuch, that two lines drawn from the extremes of the unknown ſide to any angle of the polygon, ſhall form a right-angle.



For, if you would have the polygon ABCDEF to be the greateſt poſſible, and yet ADF, ſubtended by the unknown ſide AF, not a right-angle: Then let PSO be a right-angle, contained under PS = AD, and OS = FD; and upon PS and OS, let the figures PSRQ and OST be deſcribed equilateral, and equal, to ADCB and FDE[s]. [s] 10. 6.



The triangle PSO is greater than ADF[t]; therefore, PSRQ being = ADCB, and OST = FDE[u], the whole polygon PQRSTO is alſo greater than the whole polygon ABCDEF[w], *which is repugnant.*

[t] Theor. 4.
[u] Conſtr.
[w] Ax. 6. 1

COROLLARY.

Hence, becauſe the angle in a ſemi-circle is a right-angle[x]; it appears, that the greateſt polygon that can be contained under any propoſed number of given ſides, and one other ſide any how taken, will be, when *it* may be inſcribed in a ſemi-circle, whereof the indetermined line will be the diameter.

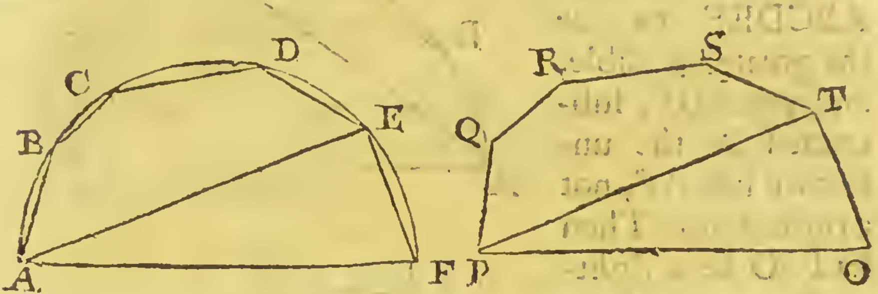
[x] 13. 3.

THEO-

THEOREM XII.

A polygon ABCDEA *in a circle, is greater than any other polygon* PQRSTP, *whatever, whose sides are the same both in length and number.*

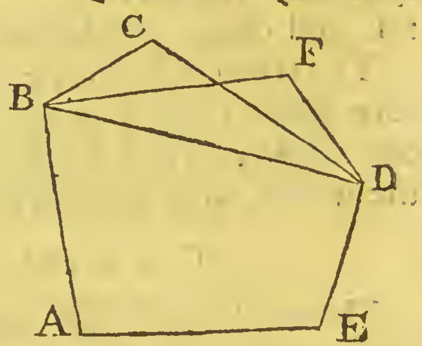
Let AF be the diameter of the circle, and join E, F; also make the angle PTO = AEF, TO = EF, and let PO be drawn.



Because AB = PQ, BC = QR, CD = RS, DE = ST, and EF = TO[y], the polygon ABCDEF, being inscribed in a semi-circle, will be greater than the polygon PQRSTO[z]; and, if from these, the equal triangles AEF, PTO be taken away, there will remain ABCDEA $\sqsubset$ PQRSTP. *Q. E. D.*

y Hyp.

z Theor. 11.



That the magnitude of the greatest polygon, which can be contained under any number of unequal sides, does not at all depend upon the order in which those lines are connected to each other, will appear, thus. Let ABCDE be the greatest, one way, or according to one order of the sides; and upon BD let a triangle BDF be constituted whose sides DF and BF are, respectively, equal to BC and DC; then, the triangles BCD, BFD being equal, the whole

whole polygons ABCDE and ABFDE will likewise be equal, notwithstanding their equal sides BC, DF, &c. are placed according to different orders.

THEOREM XIII.

Of all polygons, contained under the same perimeter, and number of sides; that whose sides, and angles, are equal, is the greatest.

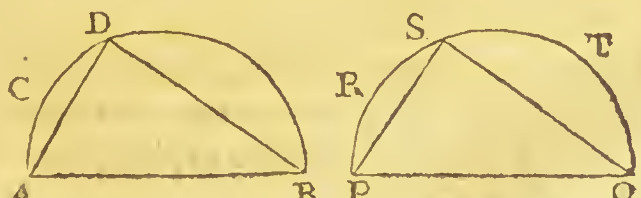
For, the greatest polygon that can be contained under a given perimeter, is one whose sides are all equal [a]. But, of all the polygons of this sort, that is the greatest which may be inscribed in a circle [b]: Therefore the greatest of all, is that whose sides are all equal, and which may be inscribed in a circle, or whereof the angles, as well as the sides, are all equal. *Q. E. D.*

[a] Theor. 10.
[b] Theor. 12.

THEOREM XIV.

The greatest area that can possibly be contained by one right line, any how taken, and any other line or lines, whatever, whereof the sum is given; will be, when two right-lines drawn from the extremes of the unknown line first mentioned, to meet any where in the given boundary, make right-angles with each other.

For, if you would have the area ACDEBA, contained by some right-line AB, and ACDEB whereof the length is given, to be the greatest possible, and ADB, at the same time, not a right angle: Then, let PSQ be a right-angle, contained under PS = AD, and QS = BD; and, having joined



PQ,

PQ, upon PS and QS conceive two figures PRS and QST to be formed, equal, and alike in all respects to ACD and DBE. Since the area PSQ is greater than ADB [c]; it is manifest, that the area PRSTQP, contained by the right line (PQ) and PRSTQ (= ACDEB [d]) will also be greater than the area ACDEBA [e], *which is repugnant:* Therefore the area ACDEBA cannot be the greatest possible, unless the angle ADB be a right one. *Q. E. D.*

[c] Theor. 4.
[d] Hyp.
[e] Ax. 6.

COROLLARY.

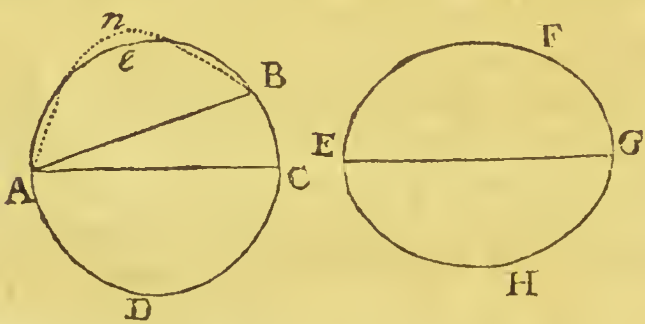
Hence, because the angle in a semi-circle is a right-angle [f], it is evident that the area will be the greatest possible, when the given length, or boundary, forms the arch of a semi-circle; whereof the indetermined right-line proposed is the diameter.

[f] 13. 3.

THEOREM XV.

Of all plane figures ABCD, EFGH, *contained under equal perimeters (or limits), the circle* (ABCD) *is the greatest.*

For, if the diameter AC be drawn, and EFG be taken equal to the arch ABC; then the area ABCA will (*by the precedent*) be greater than the



area EFGE, contained by EFG and the right-line EG; and ADCA will also be greater than EHGE: Therefore ABCD must, necessarily, be greater than EFGH. *Q. E. D.*

COROL-

COROLLARY.

Hence it appears, that the greateſt area that can poſſibly be contained by a right-line AB, and a curve-line A*e*B, both given in length; will be, when the latter is an arch of a circle. For, let A*n*B be any other curve-line, equal to A*e*B, and let the whole circle A*e*BCD be completed; which will (it is proved) be greater than the mixed figure A*n*BCD; and conſequently, by taking away the common ſegment ABCD, there will remain A*e*BA greater than A*n*BA.

THEOREM XVI.

The greateſt parallelepipedon that can be contained under the three parts of a given line AB, *any how taken, will be when all the parts are equal to each other.*

For, *if poſſible*, let two parts AE, ED

be unequal. Biſect AD in C; then will the rectangle under AE (AC+CE) and ED (AC — CE) be leſs than AC^2 (or $AC \times CD$) by the ſquare of CE [g]. Therefore the ſolid $AE \times ED \times DB$ will alſo be leſs than the ſolid $AC \times CD \times DB$ [h]; *which is contrary to hypotheſis.*

[g] 7. 2.
[h] 21. 7.

COROLLARY.

Hence, of all rectangular parallelepipedons, having the ſum of their three dimenſions the ſame, the cube is the greateſt.

THEO-

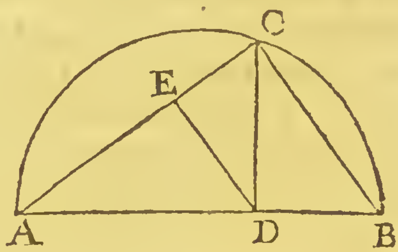
THEOREM XVII.

The greatest parallelepipedon $AC^2 \times CB$ *that can possibly be contained under the square of one part* AC *of a given line* AB *and the other part* AC, *any how taken; will be, when the former part is the double of the latter.*

For, let A*c* and B*c* be any other parts, into which the given line AB may be divided; and let AC and
i Cor. to 6. 2. and 7. A*c* be bisected in D and *d*. So shall $AC^2 \times CB = 4AD \times DC \times CB$ [i] $\sqsubset 4Ad \times dc \times cB$ [k] ($Ac^2 \times cB$ [i])
k Cor. to 6. 2. *by the precedent.* *Q. E. D.*

THEOREM XVIII.

The hypothenuse AB *of a right-angled triangle* ABC *being given; the solid* $BC \times AC^2$ *contained under one leg* BC *and the square of the other* AC, *will be the greatest possible, when the square of the latter leg* AC *is double to that of the former* BC.

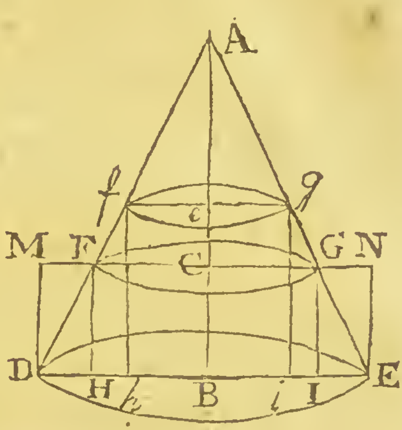


For, if CD be conceived perpendicular to AB, and DE to AC; it
l Cor. to 19. 4. will be AC^2 ($AB \times AD$ [l]) : AB^2 : : AD : AB [m] : : DE :
m 21. 7. BC [n]; and consequently
n 14. 4. o 23. 7. $AC^2 \times BC = AB^2 \times DE$ [o];
p Cor. 2. to 6. 2. which (as AB^2 is given) will, evidently, be the greatest possible, when DE, or its square [p] (DE^2) is the
q Cor. to 11. 4. greatest possible. But $DE^2 : AD^2$: : [q] BC^2 ($BD \times AB$ [l]) : AB^2 : : BD : AB [m]; and therefore $DE^2 \times AB = AD^2 \times BD$ [o]; which (and consequently DE^2) will be the greatest possible, when AD is the double
r Theor. 17. of BD [r]; that is, when AC^2 ($AD \times AB$) is the double of BC^2 ($BD \times AB$). *Q. E. D.*

THEO-

THEOREM XIX.

The altitude BC *of the greatest cylinder* HG *that can possibly be inscribed in any cone* ADE, *is one third part of the altitude* AB *of the cone, and the cylinder itself* $\frac{4}{9}$ *parts of the cone.*



For, let *gh* be any other cylinder inscribed in the cone; and it will be, $AC^2 \times BC : CG^2 \times BC :: AC^2 :$ [s] $CG^2 :: Ac^2 : cg^2$ [t] $:: Ac^2 \times Bc :$ [s] $cg^2 \times BC$; whence, *by alternation*, $AC^2 \times BC : Ac^2 \times Bc :: CG^2 \times BC : cg^2 \times Bc$: and so likewise is the cylinder HG to the cylinder [u] *hg*; but $AC^2 \times BC$ is greater [w] than $Ac^2 \times Bc$; therefore HG is also greater than *hg*. Again, since $AC = \frac{2}{3}AB$, and therefore CG [x] $= \frac{2}{3}BE$; we also have, cylinder HG : cone ADE (or cylinder [y] DN) $:: CG^2 (\frac{4}{9}BE^2) :$ [z] $BE^2 :: \frac{4}{9} : 1$. *Q. E. D.*

[s] 22. 7.
[t] Cor. 11. 4.
[u] 3. and 5. of 8.
[w] Theor. 17.
[x] Ax. 4.
[y] Cor 3. to 8. 8.
[z] 21. 7. and 3. 8.

SCHOLIUM.

From this proposition, by reasoning as in the Scholium to Theorem VIII. it will appear, that the least cone that can be described about, and the greatest cylinder that can possibly be described in, any solid generated by the rotation of a curve, concave to its axis, will be, when the sub-tangent is two-thirds of the altitude of the cone, or twice the altitude of the cylinder; and that the two figures will be in the ratio of nine to four. From whence the dimensions of the greatest and least cylinders and cones, that can be described in, and about solids generated by curves, to which the method of drawing tangents is known, may be readily determined.

THE

CONSTRUCTION

Of a great Variety of

Geometrical Problems.

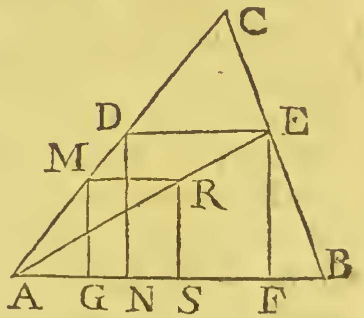
Being a farther

APPLICATION of what has been delivered in the Elementary Part of this Work.

PROBLEM I.

In a given triangle ABC, *to inscribe a square* DEFN.

CONSTRUCTION.



FROM any point M, in either side, upon the base AB, let fall the perpendicular MG; make MR perpendicular, and equal, thereto, and let ARE be drawn, meeting the other side of the triangle in E; then draw ED parallel, and EF and DN perpendicular, to AB; *and the thing is done.*

DE-

DEMONSTRATION.

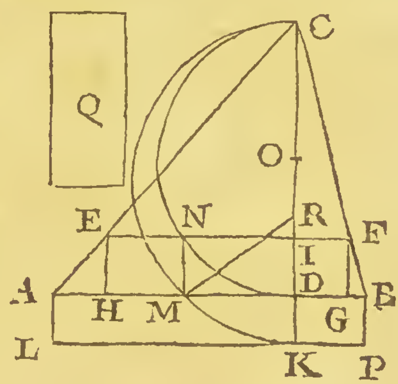
Let RS be drawn parallel to EF: Then (*by similar triangles*) RS (MG) : EF (: : AR : AE) : : MR : DE : Therefore, as MG and MR are equal, *by construction*, EF and DE will likewise be equal.

By the same method a rectangle may be inscribed in a triangle, whose sides shall be in a given ratio; if MR and MG (instead of being equal) be taken in the given ratio; the rest of the construction being exactly the same.

PROBLEM II.

In a given triangle ABC, *to inscribe a rectangle* EFGH *equal to any given right-lined figure* Q, *not exceeding half the triangle.*

CONSTRUCTION.



On the base AB (*by* 7. 6.) let a rectangle ABPL be constituted = Q; and let LP meet the perpendicular CD of the triangle (produced) in K. Then (*by* 17. 5.) let CD be divided in I, so that CI × DI = CD × DK (that is, let two semi-circles be described on CD and CK; drawing MN and NI parallel to CD and AB): So shall DI be the altitude of the required rectangle.

DEMONSTRATION

Since (*by Constr.*) $CI \times DI\ (= NI^2 = MD^2) = CD \times DK$, thence will DI : DK : : CD : CI : : AB : EF (*by* 20. 5.); and consequently DI × EF (*by* 10. 4.) = AB × DK = Q. *Q. E. D.*

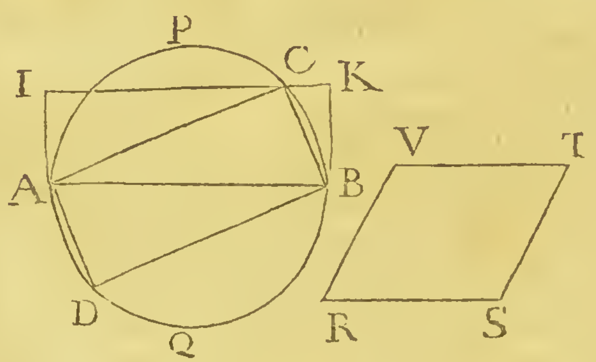
 That

That the Problem will be impossible, when Q is greater than half the triangle, is evident from the Construction, as well as from the Theorem on *p.* 200. It may also be observed, that there is another way, besides that used above, for dividing CD in the manner proposed; which (though not more obvious) is, in point of conciseness, rather preferable; and is thus. Having (*as before*) found a mean-proportional DM between CD and DK, and bisected CD in O; from M to CD apply MR $=$ OD, and take OI $=$ RD. So shall $CI \times DI = OD^2 - OI^2$ (*by* 7. 2.) $= MR^2 - RD^2$ (*by Hyp.*) $= DM^2 = CD \times DK$ (*as before*).

PROBLEM III.

In a given circle APBQ, *to inscribe a rectangle equal to a given right lined figure* RSTU, *not exceeding half the square of the diameter.*

CONSTRUCTION.



Upon the diameter AB describe the rectangle ABKI $=$ RSTU (*by* 7. 6.); and from the point C, where the side KI intersects the periphery of the circle, draw CA and CB, parallel to which draw BD and AD; then will ACBD be the rectangle that was to be constructed.

DEMONSTRATION.

The lines AC, BD, and AD, BC being parallel (*by Conſtr.*) and the angle ACB a right one (*by* 13. 3.) the figure ACBD is a rectangle (*by Cor.* to 24. 1.) and D is alſo in the circumference of the circle. But ACBD $=$ 2ACB $=$ ABKI $=$ RSTU.

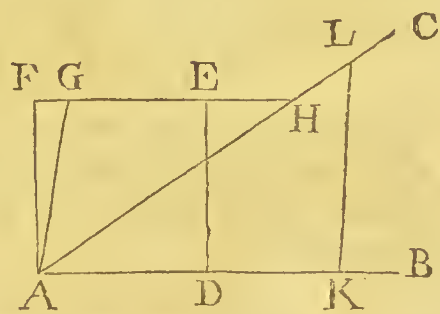
That the Problem will be impoſſible, when BK is greater than $\frac{1}{2}$AB, or when BI (RT) is greater than $\frac{1}{2}$AB^2, is manifeſt from hence, becauſe KI will then fall intirely above the circle.

PROBLEM IV.

To draw a line KL *parallel to a given line* AG, *which ſhall terminate in two other lines* AB, AC, *given by poſition, ſo as to form with them a triangle* AKL, *equal to a given rectangle* ADEF.

CONSTRUCTION.

Let FE, produced, meet AG and AC, in G and H; and, in AB, take a mean-proportional AK between GH and 2EF; then draw KL parallel to AG, *and the thing is done.*



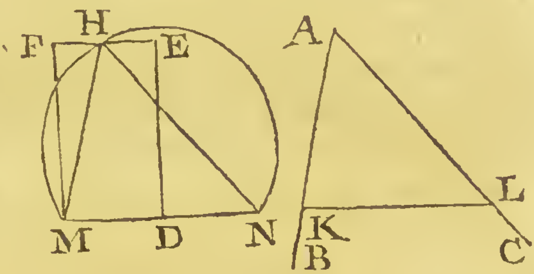
DEMONSTRATION.

The triangles AKL, HGA being equiangular, it will be AKL : HGA : : AK^2 ($=$ GH $\times$ 2EF, *by Conſtr.*) : GH^2 : : EF : $\frac{1}{2}$GH (*by* 7. 4.) : : EF $\times$ AF : $\frac{1}{2}$GH $\times$ AF ($=$ HGA) : Therefore, the conſequents being equal, the antecedents AKL and EF $\times$ AF muſt alſo be equal. *Q. E. D.*

PROBLEM V.

Between two lines AB, AC, *given by position, to apply a line* KL, *equal to a given line* MN, *so that the triangle* AKL *formed from thence, shall be of a given magnitude.*

CONSTRUCTION.

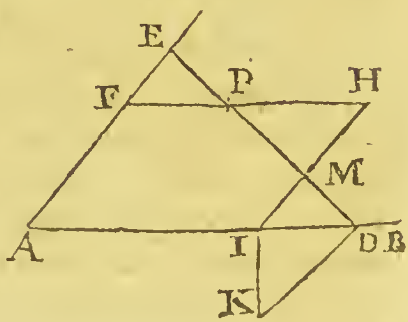


Having bisected MN in D, on MD describe a rectangle MDEF (*by* 7. 6.) = the magnitude given: also on MN let a segment of a circle be described (*by* 22. 5.) to contain an angle = A; and from its intersection with EF, draw HM and HN; then make AK = HM, AL = HN, and join K, L. So shall the base KL be also = the base MN, and the triangle AKL equal, and like in all respects, to HMN; which last (*by Cor.* to 2. 2.) is, manifestly, equal to the magnitude given MDEF. *Q. E. D.*

PROBLEM VI.

Through a given point P, *to draw a line* EPD *to meet two lines* AB, AC, *given by position, so that the triangle* ADE *formed from thence, shall be of a given magnitude.*

CONSTRUCTION.



Draw FPH parallel to AB, intersecting AC in F; and on AF, let a parallelogram AFHI be formed, equal to the given area of the triangle: Make IK perpendicular to AI, and equal to

to FP; and from K, to AB, apply KD = PH; then draw DPE, *and the thing is done.*

DEMONSTRATION.

The triangles PHM, PFE and MDI, by reaſon of the parallel lines, are ſimilar; and therefore, ſince the three homologous ſides PH (KD), FP (IK) and DI are ſuch, as to form a right-angled triangle (*by Conſtr.*), the triangle PHM on the firſt of them, is equal to both the other two FPE and MDI (*by* 29. 4.): and, if to theſe equal quantities, AFPMI be added; then will AFHI be alſo = ADE.

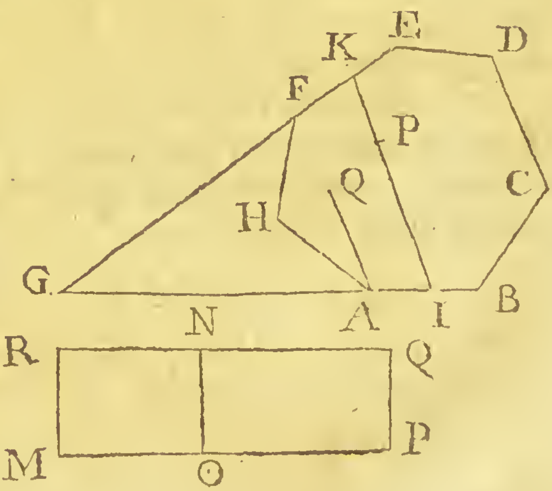
This Problem will be impoſſible, when KD (PH) is leſs than KI (PF); that is, when the area given is leſs than a parallelogram under AF and 2FP.

PROBLEM VII.

From a given polygon ABCDEFH, *to cut off a part* AIKFH, *equal to a given rectangle* MN, *by a line* (IK); *either parallel to a given line* AQ, *or paſſing through a given point* P.

CONSTRUCTION.

Let BA and EF be produced to meet in G; and upon ON let a rectangle OQ be conſtituted (*by* 7. 6.) equal to AGFH; then, by prob. the 4th, or 6th, according to the caſe propoſed,



draw IK, so as to make the triangle GKI = MQ, *and the thing is done.*

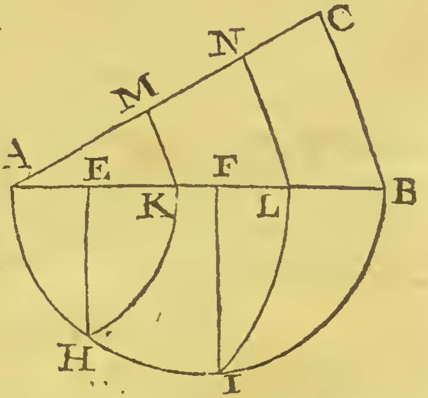
The Demonstration whereof is manifest from the Construction.

And, in the same manner, the polygon may be divided according to any given ratio; because, the whole being given, each part will be given.

PROBLEM VIII.

To divide a given triangle ABC *into any proposed number of parts* (AKM, KN, LC) *so as to have any given proportion to each other; by means of lines drawn parallel to one of the sides* BC *of the triangle.*

CONSTRUCTION.



Let AB be divided into parts, AE, EF, FB, having the same given proportion to each other, as the parts of the triangle are to have. Upon AB let a semi-circle AHIB be described; and perpendicular to AB, draw EH, FI, meeting the circumference in H and I: From the center A, through H and I, describe the arcs HK, IL, meeting AB in K and L; then draw KM and LN parallel to BC, *and the thing is done.*

DEMONSTRATION.

The triangles AKM, ALN and ABC, are in proportion, to one another, as AK^2 (AB × AE), AL^2 (AB × AF) and AB^2 (*by* 19. and 24. 4.); that is, as AE, AF and AB (*by* 7. 4.) Whence (*by division*) the proposition is manifest.

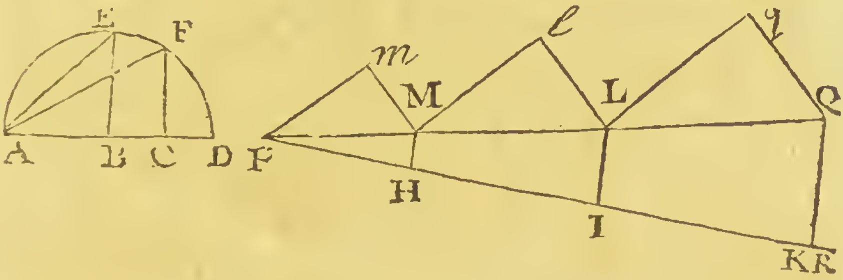
PRO-

PROBLEM IX.

To divide a given line PQ *into any propoſed number of parts, ſo that ſimilar right-lined figures* PM*m*, ML*l*, LQq, *deſcribed upon them, ſhall have the ſame given ratio among themſelves, as an equal number of right-lines* AB, AC, AD *aſſigned.*

CONSTRUCTION.

Upon the greateſt AD of the given lines AD, AC, AB, deſcribe a ſemi-circle AEFD; and perpendicular to AD, draw BE and CF, meeting the circumference in E and F; and, having drawn



PR, at pleaſure, in it take PH = diſt. AE, HI = diſt. AF, and IK = AD; draw KQ, and parallel thereto draw HM, IL; which will divide PQ, as required.

DEMONSTRATION.

PM*m* : LQ*q* : : $PM^2 : LQ^2$ (*by* 26. 4.) : : PH^2 ($= AE^2 = AB \times AD$, *by* 19. 4.) : IK^2 (AD^2) : : AB : AD (*by* 7. 6.). In the very ſame manner it appears, that ML*l* : LQ*q* : : AC : AD. *Q. E. D.*

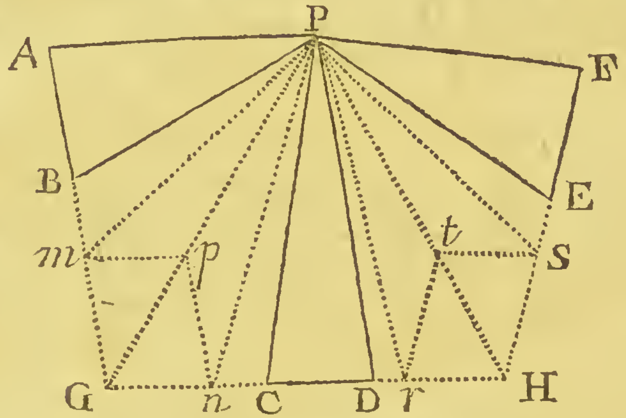
PRO-

PROBLEM X.

To determine the position of a point P, *so that lines drawn from thence to the extremes of three right-lines* AB, CD, EF, *given in length and position, shall form three triangles* ABP, CPD, EPF, *mutually equal to each other.*

CONSTRUCTION.

Let the given lines be produced to meet in G and H; in which take Gm = AB, Hs = EF, and



Gn, Hr, equal each to CD: Complete the parallelograms Gmpn, Hrts; and let the diagonals Gp and Ht be produced till they meet in P, *and the thing is done.*

DEMONSTRATION.

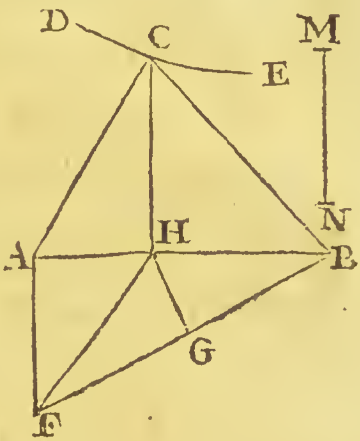
Let PA, PB, Pm, &c. be drawn. The triangles GPn, GPm, having the same base GP, and equal altitudes (because GMpn is a parallelogram) are therefore equal to each other: But CPD = GPn, and APB = GPm (*by Cor. to* 2. 2.); whence CPD and APB are likewise equal. And, from the very same reasoning, it appears that CPD and FPE are equal. *Q. E. D.*

PRO-

PROBLEM XI.

From two given points A, B, *to draw two lines* AC, BC, *to meet in a line* DE *of any kind, given by position; so that the difference of their squares shall be equal to a square given* (MN^2).

CONSTRUCTION.



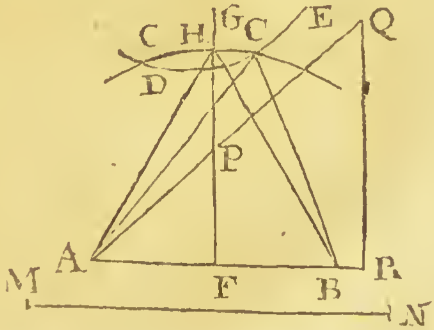
Make AF perpendicular to the line AB, and equal to MN; draw BF, which bisect with the perpendicular GH; and from its intersection with AB, draw HC perpendicular to AB, meeting DE in C; draw AC and BC, *and the thing is done.*

For $BC^2 - AC^2 = BH^2 - AH^2$ (*by* 9. 2.) $= FH^2 - AH^2 = AF^2 = MN^2$.

PROBLEM XII.

From two given points A, B, *to draw two lines* AC, BC, *to meet in a line* DE *of any kind, given by position; so that the sum of their squares shall be equal to a given square,* MN^2.

CONSTRUCTION.



Bisect AB with the perpendicular FG, in which take FP = FA, and draw APQ = MN; on AB (produced, if necessary) let fall the perpendicular QR; from A to FG apply AH = AR, and

about

about the center F, thro' H, let the circumference of a circle be described; and from its intersection C with the given line DE, draw CA and CB, *and the thing is done.*

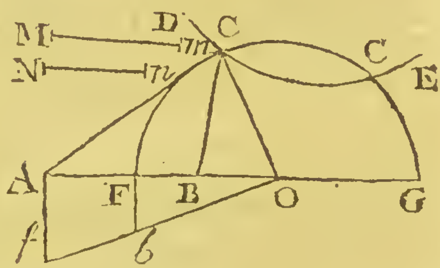
DEMONSTRATION.

Let BH be drawn; which being $= AH = AR = RQ$ (because $FP = AF$); thence will $AC^2 + BC^2 = AH^2 + BH^2$ (*by* 20. 3.) $= AR^2 + RQ^2 = AQ^2 = MN^2$. *Q. E. D.*

PROBLEM XIII.

From two given points A, B, *to draw two lines* AC, BC, *meeting in a line* DE *of any kind, given by position; so as to obtain the ratio of two unequal right-lines* Mm, Nn *assigned.*

CONSTRUCTION.



Having joined the given points, divide AB in F (*by* 15. 5.) so that $AF : BF :: Mm : Nn$; make A*f* and F*b* parallel to each other, taking the former $= AF$, and the latter $= FB$; and thro' their extremes draw *fb*O, meeting AB, produced, in O; from whence, with the radius OF, let the semi-circle FCG be described, cutting DE in C; then draw AC and BC, *and the thing is done.*

DEMONSTRATION.

Because $OA : Af\ (AF) :: OF : Fb\ (FB)$, we have (*by division*) $OA : OF :: OF : OB$; or $OA : OC :: OC : OB$. And so, the triangles OAC, OCB, having one angle FOC common, and the sides about it proportional, they must, therefore, be similar

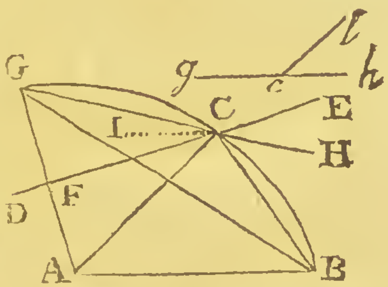
milar (*by* 15. 4); whence the other ſides will alſo be proportional, or AC : BC : : OA : OC (OF) : : Af : Fb : : Mm : Nn (*by Conſtr.*) *Q. E. D.*

Note, When, in either of the two preceding Problems, the circle deſcribed, neither cuts nor touches the given line DE, the thing propoſed to be done, will be impoſſible; as no two lines drawn from A and B, to meet above the circumference, can poſſibly have their ratio, or the ſum of their ſquares, the ſame as two lines meeting in the circumference.

PROBLEM XIV.

From two given points A, B, *to draw two lines* AC, BC, *meeting in a right-line* DE, *given by poſition*; *ſo as to make therewith two angles* ACD, BCE, *whoſe difference ſhall be equal to an angle given*, bch.

CONSTRUCTION.



Make AFG perpendicular to DE, and FG = AF; and, having drawn GB, on it let a ſegment of a circle GCB be deſcribed (*by* 22 5.) to contain an angle equal to the ſupplement (*bcg*) of the given one *bch*; and from its interſection (C) with DE, draw CA and CB; *and the thing is done.*

For, if BC and GCH be drawn, then will ACD = GCD = ECH = BCE — BCH (*bch*).

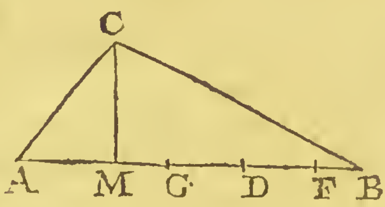
In the ſame manner, the Problem will be conſtructed, when, inſtead of the difference of ACD and BCE, *that* of ABC and BAC is given: Becauſe, when DE is parallel to AB, the latter difference

ference is equal to the former; and, in all other cases, differs from it by twice the given angle GCI, expressing the inclination of the said lines.—When the sum of the angles ACD and BCE is given, the angle ACB is also given: And here, nothing more is necessary, than barely to describe, upon AB, a segment of a circle to contain the said given angle ACB.

LEMMA.

If, of any three proportional lines AB, DB, FB, *the difference* AF *of the two extremes be bisected in* G; *and if on the greatest* AB, *as a base, a triangle* ABC *be so formed, that its lesser side* AC *shall be to the distance* MG *of the perpendicular from the bisecting point* G, *in the given ratio of* AB *to* DB; *then shall the greater side* BC *exceed the lesser* AC *by the given line* DB.

DEMONSTRATION.



Because FM exceeds AM by 2GM, BM will exceed *it* by $2GM + BF$; and the rectangle under this excess and the whole base AB ($= 2MG \times AB + BF \times AB$) will therefore (*by* 9. 2) be $= BC^2 - AC^2$. But (*by hypothesis and* 10. 4.) $MG \times AB = AC \times BD$, and $BF \times AB = BD^2$: Therefore $2AC \times BD + BD^2 = BC^2 - AC^2$; and, by adding AC^2, common, $AC^2 + 2AC \times BD + BD^2$ (or the square of $AC + BD$, *by* 6. 2.) will be $= BC^2$; and, consequently, $AC + BD = BC$. *Q. E. D.*

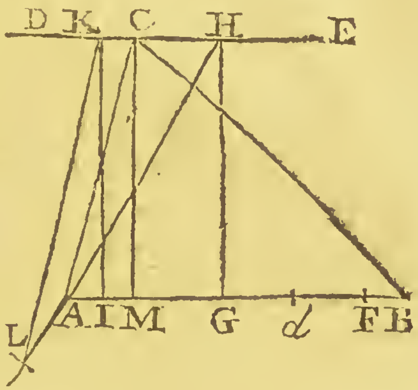
This *Lemma* is not only of use in the Problem next following, but will be found a ready instrument in the solution of many others; for which reason it is here put down.

PRO-

PROBLEM XV.

From two given points A, B, *to draw two lines* AC, BC *to meet in a right-line* DE, *given by position; so that their difference shall be equal to a given right-line* BD.

CONSTRUCTION.



In AB, take a third-proportional BF to BA and Bd; and, having bisected AF in G, take GI = Bd; make GH and IK perpendicular to AB, meeting DE in H and K; and draw HAL, to which, from K, apply KL = AB; and parallel thereto draw AC, meeting DE in C; join B, C; *and the thing is done.*

DEMONSTRATION.

Let CM be perpendicular to AB.

Then, because of the parallel lines, it will be AC : KL (AB) : : HC : HK : : GM : GI (Bd.) Whence *(by alternation)* AC being to GM, as BA to Bd; and AB, dB, FB being also proportionals; the whole Construction is manifest *from the Lemma* premised.

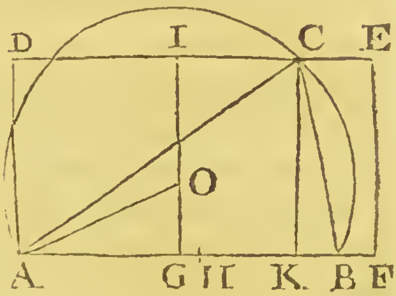
If the sum, instead of the difference, of the two lines (AC, BC) be given, the method of construction will be exactly the same, without the least alteration of any one step; provided that Bd be first of all taken (in BA, produced) equal to the given sum, instead of the difference.

PRO-

PROBLEM XVI.

From two given points A, B, *to draw two lines* AC, BC, *ſo as to meet in a right-line* DE *parallel to that* (AB) *joining the ſaid points, and that the rectangle* (AC × BC) *contained by them, ſhall be equal to a rectangle given,* ADEF.

CONSTRUCTION.



Biſect AB and AF, in G and H; and, having drawn GI perpendicular to AB, to it, from A, apply AO = AH; and from the center O, through A and B, let the circumference of a circle be deſcribed interſecting DE in C; from whence draw CA and CB, *and the thing is done.*

For, if CK be drawn perpendicular to AB, it is evident (*by* 25. 3) that AC × BC = CK × 2AO = AD × AF. *Q. E.* D.

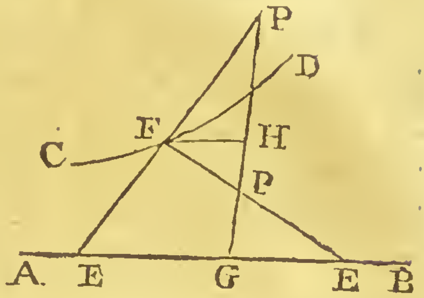
PROBLEM XVII.

Through a given point P, *ſo to draw a line* FPE, *that the parts thereof* PF, PE, *intercepted by that point and two lines* AB, CD, *given in poſition, ſhall obtain a given ratio.*

CON-

CONSTRUCTION.

Through P, to any point in AB, draw PG, in which (*by* 13. 5.) take PH to PG, in the given ratio of PF to PE; draw HF parallel to AB, meeting CD in F; then draw FPE, *and the thing is done.*



For, the triangles PGE, PHF being equiangular (*by* 3. *and* 7. 2.) thence is PF : PE : : PH : PG : and ſo PF and PE (as well as PH and PG) are in the ratio given.

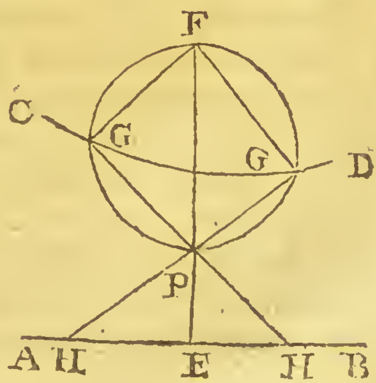
In this conſtruction, it is neceſſary that one of the two given lines ſhould be a *right-one:* The other may, it is manifeſt, be a line of *any kind* whatever.

PROBLEM XVIII.

Through a given point P, *to draw a line* GH *terminating in a right-line* AB, *and in a line* CD *of any kind, given both by poſition; ſo that the rectangle under the two parts thereof* PG, PH, *ſhall be of a given magnitude.*

CONSTRUCTION.

Having drawn EPF perpendicular to AB, take therein PF (*by* 7. 6.) ſo that PE $\times$ PF = the magnitude given: Upon PF, as a diameter, let a circle be deſcribed, interſecting CD in G; then draw GPH, *and the thing is done.*



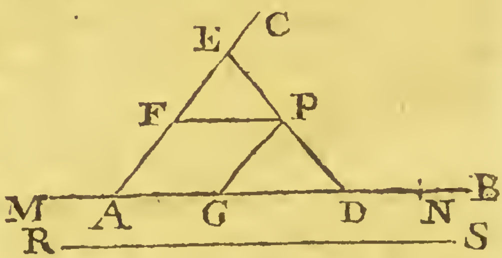
DEMONSTRATION.

If FG be drawn, the triangles PFG and PHE, having FPG = HPE, and FGP (= a right-angle, *by* 13. 3.) = PEH, will be equiangular: And, consequently (*by* 24. 3.) PG × PH = PF × PE = the magnitude given. *Q. E. D.*

PROBLEM XIX.

Through a given point P, *between two right-lines* AB, AC, *given by position, so to draw a line* ED, *that the sum of the segments* (AD + AE) *cut off by it from the two former, shall be equal to a given line* RS.

CONSTRUCTION.



Draw PF and PG parallel to AB and AC; in BA produced take AM = AF, and MN = RS; then divide GN in D (*by* 17. 5.) so that GD × ND = AM × AG; draw DPE, *and the thing is done.*

DEMONSTRATION.

By similar triangles, GD : GP (= AF = AM) : : FP (= AG) : FE; and consequently GD × FE = AM × AG = GD × ND (*by Constr.*): Whence FE = ND; and therefore FE + AF + AD (AE + AD) = ND + AM + AD = MN = RS. *Q. E. D.*

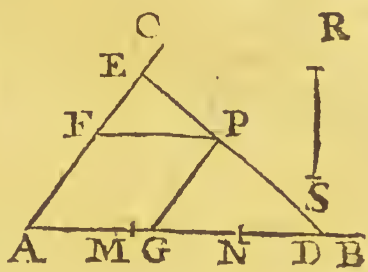
It appears from the Construction, that the Problem will be impossible, when (GN) the excess of RS above PF and PG, is less than the double of a mean-proportional between these two quantities.

PRO-

PROBLEM XX.

Through a given point P, *between two right-lines* AB, AC, *given by poſition, ſo to draw a line* DE, *that the difference* (AD — AE) *of the ſegments cut off by it from the two former, ſhall be equal to a given line* RS.

CONSTRUCTION.



Draw PF and PG parallel to AB and AC; in AB take AM = AF, and MN = RS; then (*by* 18. 5.) let a line ND be added to GN, ſo that GD × ND, PG × PF; draw DPE, *and the thing is done.*

DEMONSTRATION.

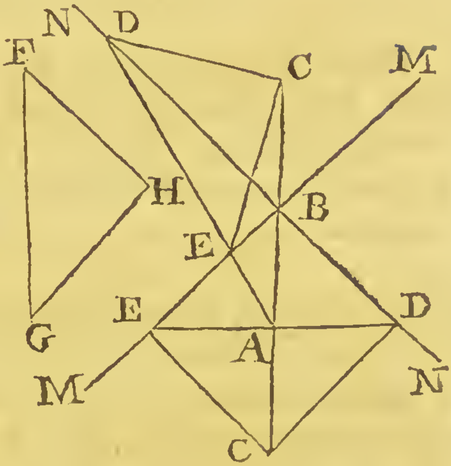
Becauſe of the ſimilar triangles PGD, EFP, we have GD × FE = PG × PF = GD × ND (*by Conſtr.*) and conſequently FE = ND; whence AD — AE = AN — AF (AM) = MN = RS. *Q. E. D.*

PROBLEM XXI.

Between two right lines NBN, MBM, *given by poſition, to apply a line* DE *of a given length, and which (produced, if neceſſary) ſhall paſs through a given point* A, *equally diſtant from the ſaid given lines.*

CONSTRUCTION.

Let FG be the length given; and, having drawn AB, make the angles F, G equal each to ABD (or ABE); and in AB produced take AC (*by* 18. 5.) so that $AC \times BC = HG^2$: from C to NN apply $CD = HG$; then draw DAE (DEA), *and the thing is done.*



DEMONSTRATION.

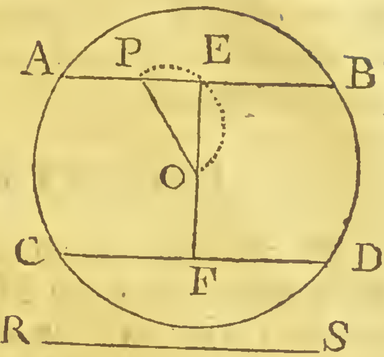
Becaufe AC : CD : : CD : BC (*by Conftr. and* 10. 4.) the triangles CAD and CDB (having one angle common, and the fides about it proportional) will be equiangular (*by* 15. 4.): And therefore, fince CDA = DBC = ABE (*by hypothefis*), the circumference of a circle may be defcribed through all the four points C, D, B, E (*by* 11. 3. *or* 19. 3.). And fo the angle DEC, ftanding on the fame fubtenfe (DC) with DBC, will be equal to it; and, confequently, equal to EDC. Therefore the ifofceles triangles EDC, FGH being equiangular, and having CD = HG, their bafes ED and FG will alfo be equal. *Q. E. D.*

PROBLEM XXII.

In a given circle ABDC, *to apply a chord* AB *equal to a given line* RS (*lefs than the diameter*) *and which fhall pafs through a given point* P.

CON-

CONSTRUCTION.



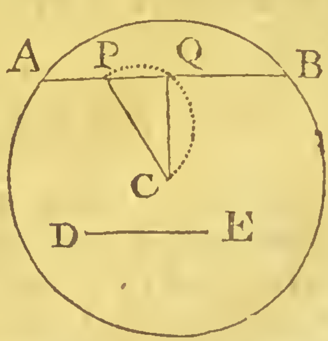
Inſcribe CD = RS, upon which, from the center O, let fall the perpendicular OF; alſo draw OP, upon which let a ſemi-circle be deſcribed, and in it apply OE = OF; laſtly, through P and E, let AB be drawn; which will, manifeſtly, be equal to CD (= RS, *by* 3. 3) as being equally diſtant from the center, by conſtruction.

When the point given is placed without the circle, the conſtruction will be no-ways different.

PROBLEM XXIII.

Through a given point P, *ſo to draw a line* AB, *that the parts thereof* AP, BP, *intercepted by that point and the circumference of a given circle, ſhall have a given difference,* DE.

CONSTRUCTION.

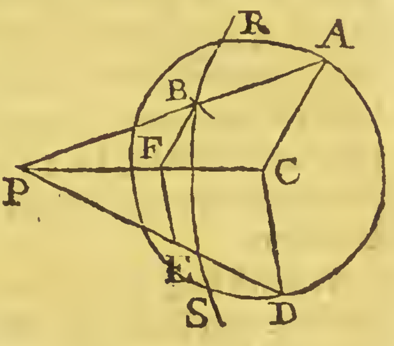


From the center C, draw CP, upon which let a ſemi-circle PQC be deſcribed, and in it apply PQ equal to half DE, producing the ſame, both ways, to meet the circumference in A and B: So ſhall AQ = BQ (*by* 2. 3.); and therefore PB (= BQ + PQ = AQ + PQ = AP + 2PQ) = AP + DE, *which was to be done.*

PROBLEM XXIV.

From a given point P, *to the circumference of a given circle* C, *to draw a right-line* PBA, *so as to be divided in a given ratio by a line* RBS *of any kind, given in position.*

CONSTRUCTION.



To any point D in the circumference, draw PD, which divide at E in the ratio given; and, having drawn PC and the radius CD, parallel to the latter draw EF, meeting the former in F; from whence, to RS, apply FB = FE; then through B draw PA, *and the thing is done.*

DEMONSTRATION.

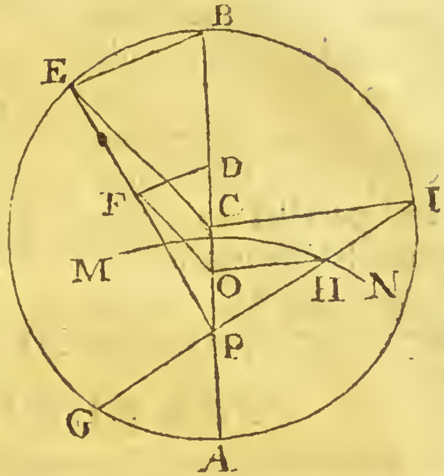
Let CA be drawn. Then, because of the parallel lines CD, EF, it will be, CD (AC) : EF (FB) : : PC : PF; and so the triangles PAC, PBF will likewise be equiangular (*by* 16. 4.). And therefore PB : BA : : PF : FC : : PE : ED (*by* 13. 4). *Q. E. D.*

PROBLEM XXV.

Through a given point P, *to draw a line* GH, *terminating in the circumference of a given circle* BGA, *and in a line* MN *of any kind, given by position; so that the rectangle under the two parts thereof* PG, PH, *shall be of a given magnitude.*

CON-

CONSTRUCTION.



Through P draw the diameter AB, in which let PD be so taken (*by* 7. 6.) that PA × PD = the magnitude given. From any point E in the circumference, draw EP and the radius EC; and, having joined BE, draw DF parallel to BE, and from its intersection with PE, draw FO parallel to EC, meeting PB in O; from whence, to MN, apply OH = OF; and through P draw HG for the line required.

DEMONSTRATION.

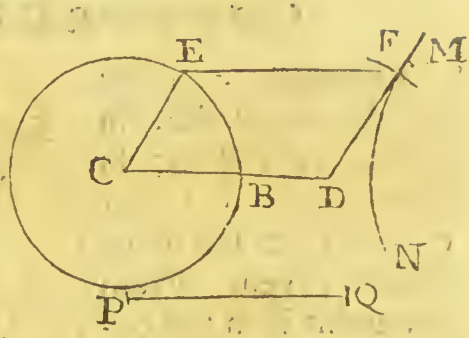
Let PH be produced (if necessary) to meet the circumference of the circle in I; and let C, I be joined.

The lines OF, CE being parallel, thence will PO : PC : : OF (OH) : CE (CI); and therefore OH and CI will likewise be parallels (*by* 16. 4.) Therefore PH : PI : : PO : PC : : PF : PE : : PD : PB; whence, alternately, PH : PD : : PI : PB : : PA : PG (*by* 21. 3. *and* 10 4.) and consequently PH × PG = PD × PA. *Q. E. D.*

PROBLEM XXVI.

From the circumference of a given circle C, *to a line* MN *of any kind, given by position, so to draw a right-line* EF, *as to be both equal and parallel to a given right line* PQ.

CONSTRUCTION.

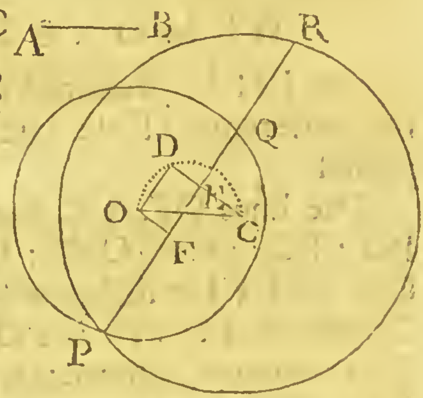


From the center C, draw CD equal and parallel to PQ; and, from D to MN, apply DF = the radius CB; draw CE equal and parallel to DF; then, EF being drawn, it will (*by* 26. 1.) be equal, and parallel to CD; *which was to be done.*

PROBLEM XXVII.

From the point of intersection P *of two given circles* O *and* C, *so to draw a line* PR, *that the part thereof* QR *intercepted by the two peripheries, shall be equal to a given line* AB.

CONSTRUCTION.



Upon the line OC joining the two centers, let a semi-circle ODC be described, in which apply OD = $\frac{1}{2}$AB; and parallel thereto, draw PR, *and the thing is done.*

DEMONSTRATION.

Let CD, and OF parallel to CD, be drawn, meeting PR in E and F. Then, the angle ODC being a right one (*by* 13. 3.) and PR parallel to DO, E and F will also be right-angles, and EF = DO (*by* 24. 1.): And so, PE — PF being (= DO) = $\frac{1}{2}$AB, it is manifest, that 2PE (PR) — 2PF (PQ) = AB, or that QR = AB. *Q. E. D.*

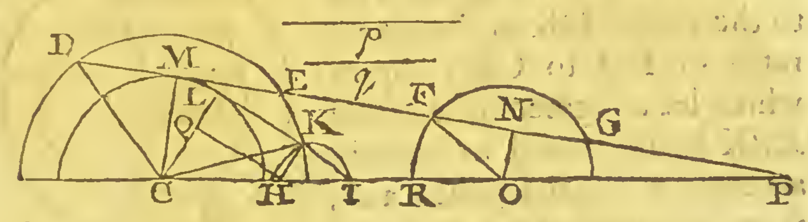
PRO-

PROBLEM XXVIII.

From a given point P, *in the line passing through the centers of two given circles* C *and* O; *so to draw a line* PD, *that the parts thereof* DE, FG, *intercepted by those circles, shall be in a given ratio,* (viz. *as* p *is to* q).

CONSTRUCTION.

Take CH to OR, as *p* to *q*; and CI to OR, as PC to PO: Upon HI let a semi-circle be described, intersecting the circle CDE in K; through



which point draw IKL, and make CL perpendicular thereto; at which distance, from the center C, let a circle ML be described: Then, if from P a line PD be drawn to touch this circle, *the thing is done.*

DEMONSTRATION.

Let CD, CK, HK, FO be drawn, and also CM and ON, perpendicular to PD, and HQ to CL.

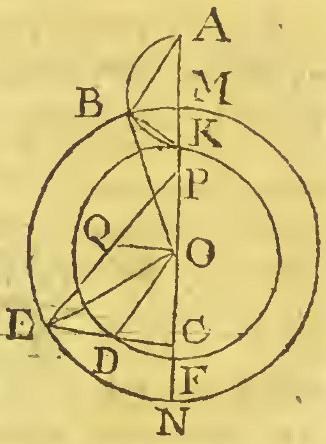
The right angled triangles CDM, CKL having CD = CK, and CM = CL, have also DM = LK = QH (since, *by* 13. 3. the angle HKI, as well as L, is a right one). Moreover (*by construction*) CI : OR (OF) : : PC : PO : : CM (CL) : ON; and so the triangles CIL, OFN (having their sides proportional) must be similar; and consequently OFN also similar to CHQ: whence, as QH (or DM) : FN : : CH : FO (OR) : : *p* : *q* (*by Constr.*). *Q. E. D.*

PRO-

PROBLEM XXIX.

To draw a line EC, *to make given angles with a line* MN *passing through the common center* O, *of two given circles* MEN, KDF, *so that the parts thereof* CD, ED, *intercepted by that line and the two peripheries, shall obtain a given ratio.*

CONSTRUCTION.



Let QOM be the given angle, to which ECM shall be equal: In OM produced, let KA be so taken, as to be to the radius OK in the given ratio of ED to CD; upon which let a segment of a circle ABK be described to contain an angle equal to QOM; and, from its intersection with the circle MEN, draw BA; parallel to which, draw the radius OD; and then, through D, draw EC parallel to QO.

DEMONSTRATION.

Let BK, BO, EO be drawn, and also EP parallel to DO, meeting OQ and OM in Q and P.

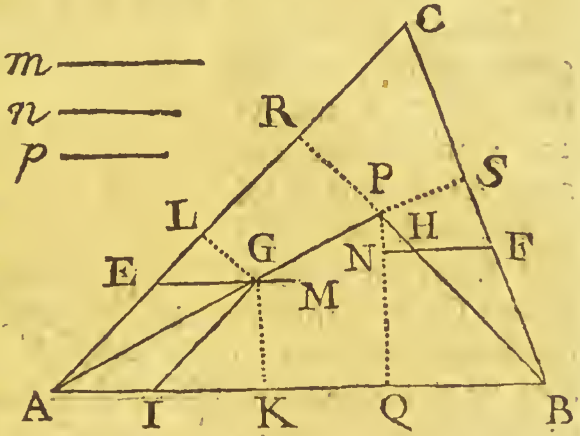
Then, the triangles AKB, POQ, having ABK = POQ (*by Construction*), and KAB = OPQ (*by* 7. 1.) have their external angles OKB, EQO also equal (*by* 9. 1.): Therefore, because EQ (= OD) = OK, and EO = BO, thence is OQ = KB (*by* 17, 1.); and therefore, as the triangles POQ, ABK are equiangular, PQ is also = AK: But (because of the parallel lines) CD : ED : : DO (OK) : PQ (AK). *Q. E. D.*

PRO-

PROBLEM XXX.

To determine a point P, *so that three perpendiculars drawn from thence, to as many right lines* AB, AC, BC, *given by position, shall obtain the ratio of three given lines* m, n *and* p, *respectively.*

CONSTRUCTION.



Take AE and BF each = *m*, and draw EM and FN parallel to AB; in which take EG = *n*, and FH = *p*; thro' G and H draw AP and BP, and the point of concourse P, *will be that which was to be determined.*

DEMONSTRATION.

Upon the sides of the triangle ABC let fall the perpendiculars GL, GK, PR, PQ, PS; and let GI, parallel to AC, be drawn.

The angle LEG being = LAK = GIK *(Cor.* 1. *to* 7. 1.*)* and L and K being both right-angles (*by Constr.*), the triangles EGL and GIK are similar; and therefore IG (*m*) : EG (*n*) : : GK : GL : : PQ : PR (*by* 21. 4.). And, in the very same manner, *m* : *p* : : PQ : PS. *Q. E. D.*

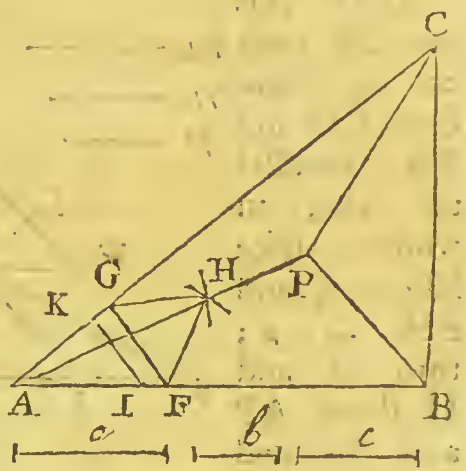
After the same way, the Problem may be constructed, when the lines drawn from P are required to make any given angles with the lines upon which they fall.

PRO-

PROBLEM XXXI.

To determine a point P, *so that three lines* PA, PB, PC, *drawn from thence to three given points* A, B, C; *shall obtain the ratio of three given lines* a, b *and* c, *respectively.*

CONSTRUCTION.



Having joined the given points, in AB take AF = *a*, and AI = *c*; make the angles AFG and AIK equal, each, to ACB; and from the centers F and G, with the radii *b* and AK, let two arcs be described, intersecting in H; from which point draw HF and HA; then draw BP to make the angle ABP = AHF, and it will meet AH (produced) in the point P, required.

DEMONSTRATION.

Let BP, CP, and GH be drawn.

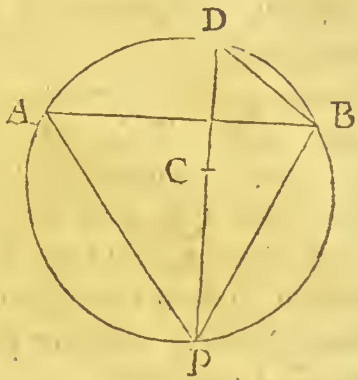
The triangles ABP, AHF being equiangular, it will be, AP : BP : : AF (*a*) : FH (*b*); also AB : AP : : AH : AF; and AB : AC : : AG : AF (because ABC and AGF are likewise equiangular, (*by Constr.*). Now, seeing the extremes of the two last proportions are the same, the four means AP; AC, AG, AH (*by* 10. 4.) will therefore be proportionals; and so, the triangles ACP, AHG being equiangular (*by* 15. 4.) it will be AP : CP : : AG : GH (AK) : : FA (*a*) : AI (*c*). *Q. E. D.*

PRO-

PROBLEM XXXII.

To determine the position of a point P, *at which, lines drawn from three given points* A, B, C, *shall make given angles, one with another.*

CONSTRUCTION.



Draw AB, upon which *(by* 22. 5.) let a segment of a circle APB be described, capable of containing the given angle which the lines drawn from A and B are to include; and let the whole circle be completed; make the angle ABD equal to that which the lines drawn from A and C are to include; and from the point D, where BD meets the circumference, through C, let DP be drawn, meeting the circumference in P; *which is the point required.*

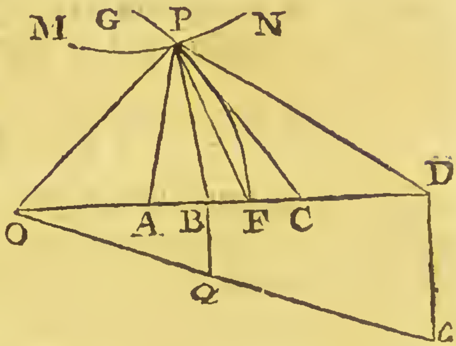
For, AP and BP being drawn, the angle APC will be = the given angle ABD *(by* 11. 3.) both standing on the same arch AD:—And APB is also of the given magnitude by construction.

PROBLEM XXXIII.

Any two unequal segments AB, CD *of a right-line* AD *being given, as well in position as length; to determine a point* P *in a line* MN *of any kind, given by position; at which the two angles* APB, CPD, *subtended by those segments, shall* (*if possible*) *be equal to each other.*

CON-

CONSTRUCTION.



Make B*a* and D*c* parallel to each other, taking the former = BA, and the latter = DC; and thro' their extremes draw *ca*O, meeting DA produced, in O: Take OF a mean-proportional between OB and OC; and from the center O, with the interval OF, let a circle FG be deſcribed; which (when the Problem is poſſible) will cut (or touch) MN, and the point of interſection P, will be *that* required.

DEMONSTRATION.

Let PO, PA, PB, PF, PC, and PD be drawn.

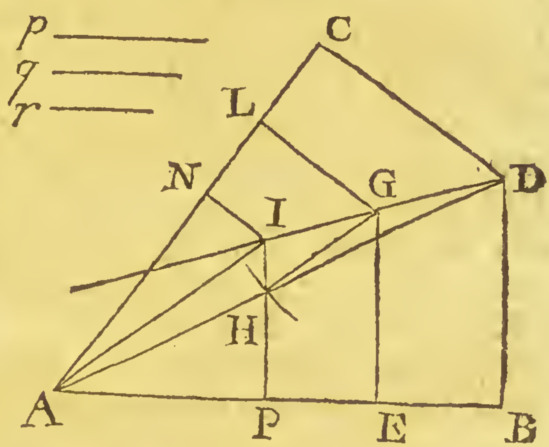
Becauſe OD : D*c* (DC) : : OB : B*a* (BA), it will be (*by diviſion*) OD : OC : : OB : OA; and conſequently $OD \times OA = OC \times OB = OP^2$ (*by Conſtr.*). Therefore, ſeeing that OA : OP : : OP : OD, and that the angle O is common to both the triangles OAP, OPD, theſe triangles muſt be equiangular (*by* 15. 4.) and conſequently APF = OPF (OFP) — OPA (ODP) = DPF (*by* 9. 1.). In the very ſame manner, becauſe OB : OP : : OP : OC, the angle BPF will be = CPF; and, conſequently, APB alſo = CPD. *Q. E. D.*

PRO-

PROBLEM XXXIV.

Two right-lines AB, AC *being given, both in length and poſition; from the point of their concourſe* A, *ſo to draw another line* AI, *that two perpendiculars* IP, IN, *falling from the extreme thereof upon the two given lines, ſhall cut off alternate ſegments* BP, CN *in a given ratio each to the line* AI *ſo drawn.*

CONSTRUCTION.



Let the given ratio of AI, BP and CN be that of the lines p, q and r, reſpectively. Take BE $= q$, and CL $= r$; making BD, EG, CD and LG perpendicular to AB and AC, ſo as to meet in D and G: draw DA and DG, and from G to AD, apply GH$=p$, and parallel thereto draw AI, meeting DG produced (if needful) in I, *and the thing is done.*

DEMONSTRATION.

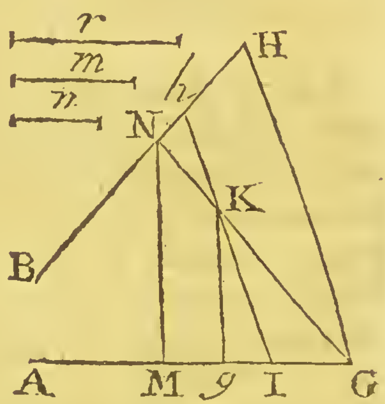
Becauſe of the parallel lines, AI : GH (: : ID : GD) : : BP : BE; whence, alternately, AI : BP : : GH (p) : BE (q). In the very ſame manner AI : CN : : p : r. *Q. E. D.*

PRO-

PROBLEM XXXV.

Between two lines AG, BH, *given both in position and length, to draw a line* MN, *which shall be in a given ratio to each of the segments* MG, NH *cut off from the two given lines.*

CONSTRUCTION.



Let the given ratio of MN, MG and NH be that of r, m and n, respectively: In AG and BH take Gg = m, and Hh = n; and, having drawn GH, parallel to it draw hI; to which, from g, apply gK = r; draw GKN, meeting BH in N, and parallel to Kg draw MN, *and the thing is done.*

DEMONSTRATION.

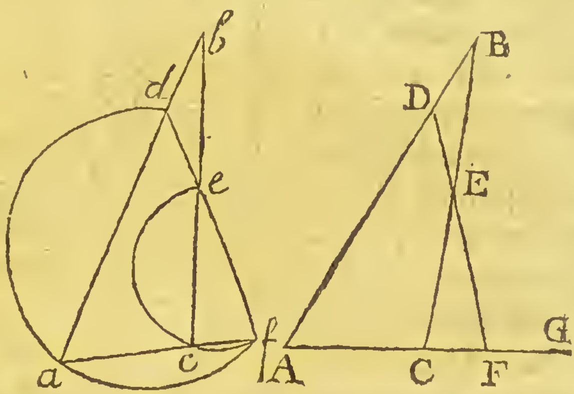
Because of the parallel lines, it will therefore be MN : GM : : gK (r) : Gg (m); and likewise MN : gK (r) : : GN : GK : : NH : Hh (n); which last (*by alternation*) is MN : NH : : r : n. *Q. E. D.*

PRO-

PROBLEM XXXVI.

To draw a line DF, *to cut three other lines* AB, AG, BC, *given by position; so that the two parts thereof* DE, EF, *intercepted by those lines, shall be respectively equal to two given lines,* de, ef.

CONSTRUCTION.



Upon the right-lines *df*, *ef* let two segments of circles, *daf*, *ecf*, be described, to contain angles, respectively equal to BAG and BCG: Then (*by Prob.* 27.) draw *fa*, so that the part *ac*, intercepted by the two peripheries, shall be equal to AC; join *a*, *d*, and take AF = *af*, AD = *ad*; then draw DF, *and the thing is done.*

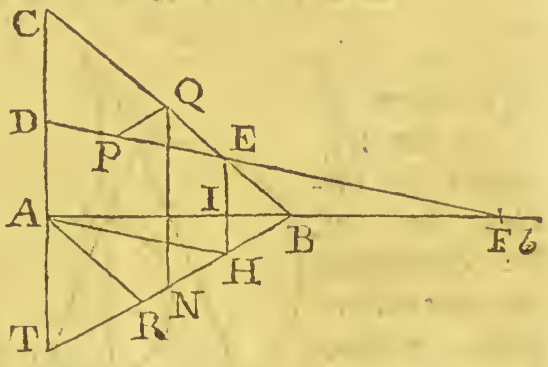
DEMONSTRATION.

The triangles FAD, *fad*, having AF = *af*, AD = *ad*, and A = *a* (*by Construction*), are equal in all respects; and therefore (if *ce* be drawn) the triangles FCE, *fce*, having F = *f*, FCE = *fce*, and CF = *cf*, will also be equal and alike: Therefore, seeing the wholes DF, *df*, and the parts FE, *fe* are equal, the remaining parts DE and *de* must likewise be equal. *Q. E. D.*

PROBLEM XXXVII.

Through a given point P, *to draw a line* DPF, *to cut three lines* AC, CB, ABb, *given by poſition*; *ſo that the parts thereof* DE, EF *intercepted by thoſe lines, ſhall obtain a given ratio.*

CONSTRUCTION.



In CA produced, take AT to CA in the given ratio of DE to EF; and, having joined B, T, draw PQ parallel thereto; and from its interſection with BC, draw QN parallel to CT; alſo draw AR parallel to BC: And in RB take RH (*by* 18. 5.) ſo that $NH \times RH = PQ \times TR$: Then draw AH, and DPEF parallel to it, *and the thing is done.*

DEMONSTRATION.

Becauſe (*by Conſtr. and* 10. 4.) NH : TR : : PQ : RH : : QE : AR (*by* 14. 4.) it follows (*by alternation*) that NH : QE : : TR : AR : : TB : BC (*by* 12. 4.) : : BN : BQ: Therefore EH (when drawn) will be parallel to QN or AD (*by Cor. to* 12. 4.); and ſo DEHA being a parallelogram, we have again (by ſimilar triangles) as DE (AH) : EF : : IH : IE : : AT : AC. *Q. E. D.*

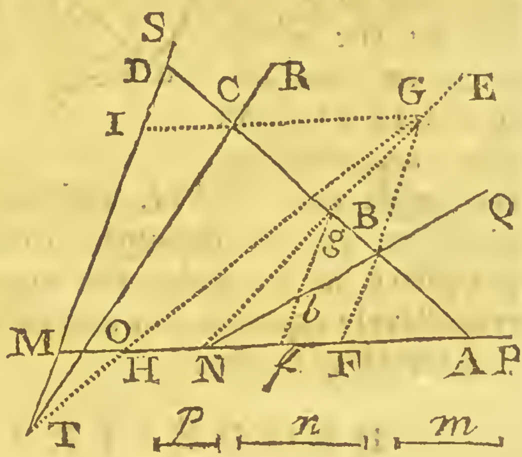
When the ſegments AD, BE, cut off from the given lines AC, BC, are required to be in a given ratio (inſtead of DE and EF), the conſtruction will be

be the same; provided that CT be taken to BC in the ratio given. For, then AD (EH) : BE : : CT : BC.

PROBLEM XXXVIII.

To draw a line ABCD *to cut four other lines* MONP, NQ, ROT, SMT, *given by position; so that the three parts thereof* AB, BC, CD, *intercepted by those lines, may obtain the ratio of three given lines* m, n, *and* p, *respectively.*

CONSTRUCTION.



From any point *f* in NP draw *fbg* parallel to TMS, intersecting NQ in *b*; and, having taken *bg* to *bf* in the given ratio of *n* to *m*, draw N*g*E: Moreover, having taken OH to MO in the given ratio of *n* to *p*, draw THG; and from (G) its intersection with NE, draw GI and GF parallel to MP and ST, respectively, cutting NQ and TR in B and C; thro' which two points draw ABCD, *and the thing is done.*

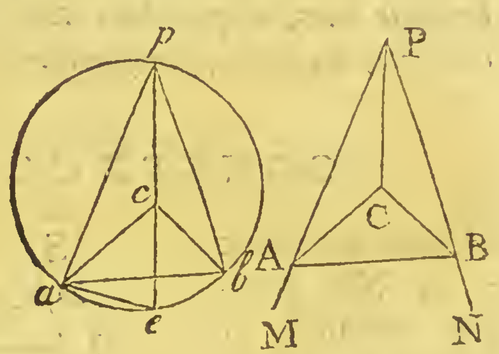
DEMONSTRATION.

By similar triangles { AB : BC : : BF : BG : : *bf* : *bg* : : *m* : *n*;
BC : CD :: CG : CI : : OH : MO :: *n*:*p*.
Q. E. D.

PROBLEM XXXIX.

To draw two lines CA, CB, *from a given point* C, *to terminate in two other lines* PM, PN, *given by position, and which, together with the line* AB, *joining their extremes, shall form a triangle,* ABC, *similar to a given one,* abc.

CONSTRUCTION.



Upon *ab* let a segment of a circle *apb* be described, to contain an angle equal to MPN, and let the whole circle be completed; draw PC, and also *ae*, making the angle *bae* = CPN, and intersecting the periphery in *e*; and through *c* draw *ep*, meeting the periphery in *p*; make the angles PCA and PCB respectively equal to *pca and pcb*; then join A, B, *and the thing is done.*

DEMONSTRATION.

If *pa* and *pb* be drawn; then will the angle *bae* (CPB) = *cpb*, both standing on the same arch *be*; therefore, APB being also = *apb* (*by Constr.*), the remaining angles APC and *apc* must consequently be equal; whence, as PCA = *pca*, and PCB = *pcb* (*by Constr.*), the triangles APC, *apc*, and BPC, *bpc* are equiangular; and therefore AC : *ac* (: : PC : *pc*) : : CB : *cb*. And so the triangles ABC, *abc*, having the sides about the equal angles ACB, *acb*, proportional, they are like to each other (*by* 15. 4.). *Q. E. D.*

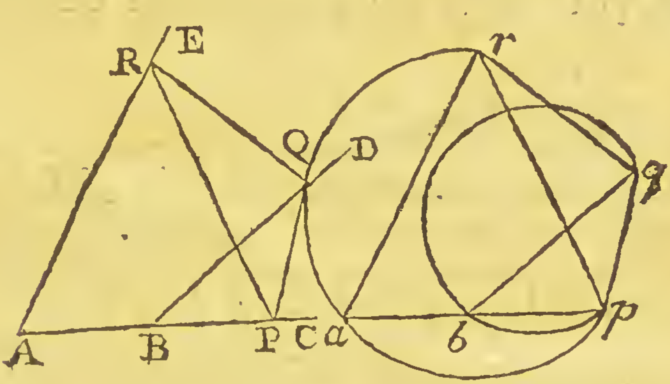
PRO-

PROBLEM XL.

To deſcribe a triangle PQR, *equal and ſimilar to a given triangle* pqr, *which ſhall have its angular points placed in three right-lines* ABC, BD, AE, *given by poſition.*

CONSTRUCTION.

Upon the two ſides *pq*, *pr*, let two ſegments of circles *pbq*, *par* be deſcribed, to contain angles reſpectively equal to CBD and CAE: Then draw *pa* (*by Prob.* 27.) ſo that the part thereof *ba*, intercepted by the two peripheries, ſhall be equal to



BA; and, having joined *b*, *q*, and *a*, *r*, make BP $=$ *bp*, BQ $=$ *bq*, AR $=$ *ar*, and let PQ, PR, and QR be drawn for the ſides of the triangle required.

DEMONSTRATION.

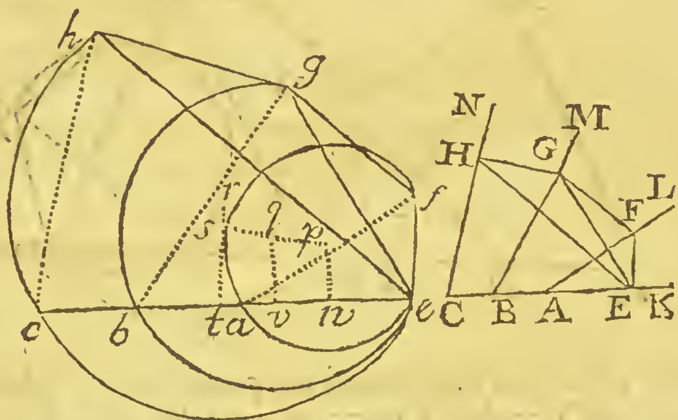
The triangles PAR, *par*; PBQ, *pbq*, being equal and alike in all reſpects (*by Conſtr.*), not only the ſides PR, *pr*; PQ, *pq*, but the angles QPR, *qpr*, will be equal; and, conſequently, the two triangles PQR, *pqr* alſo equal and like to each other. *Q. E. D.*

PROBLEM XLI.

To describe a trapezium similar to a given one efgh, *having its angular points placed in four right-lines* CN, BM, AL, CBAK *given by position.*

CONSTRUCTION.

Let three points *p*, *q*, *r* be found, from whence, as centers, segments of circles may be described, upon *ef*, *eg*, *eh*, to contain angles equal to the three given angles KAL, KBM, KCN, respectively; draw *pq*, in which, produced (if necessary) take *qs* to *qp* in the proportion of BC to AB;



and, having drawn *rst*, make *ec* perpendicular thereto, intersecting the three circles in *a*, *b*, *c*. Take AE : AB : : *ae* : *ab*; and make the angles CEH, CEG, and CEF, respectively equal to *ceh*, *ceg*, and *cef*; then let H, G and G, F be joined; and, I say, the trapezium EFGH will be similar to the given one *efgh*.

D E-

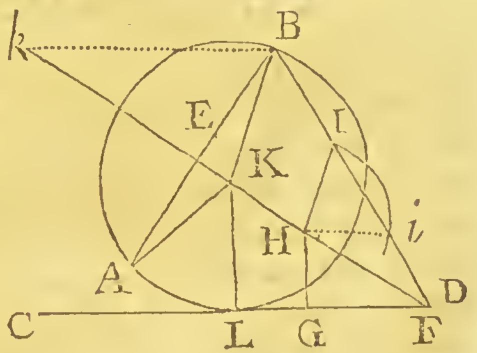
DEMONSTRATION.

Let af, bg, ch, be drawn, and alſo pw, qv, perpendicular to ec. So ſhall $ab = eb - ea = 2ev - 2ew = 2vw$; and $bc = ec - eb = 2et - 2ev = 2vt$: And therefore $ab : bc :: vw : vt :: pq : qs$ (*by* 13. 4.) :: AB : BC (*by Conſt.*). Again, ſeeing the triangles EAF, eaf; EBG, ebg; ECH, ech are, reſpectively, equiangular (*by Conſtr.*) it will be EG : eg :: EB : eb :: AE : ae (*by Conſtr. and Compoſition*) :: EF : ef (*by* 14. 4.); and ſo the triangles GEF, gef (having the ſides about the equal angles proportional) are ſimilar. And, in the very ſame manner, the two remaining triangles EGH, egh (and conſequently the whole trapeziums EFGH, $efgh$) will appear to be ſimilar. *Q. E. D.*

PROBLEM XLII.

To deſcribe the circumference of a circle through two given points A, B, *which ſhall touch a right-line* CD *given by poſition.*

CONSTRUCTION.



Draw AB, which biſect in E by the perpendicular EF, meeting CD in F; from any point H in EF, draw HG perpendicular to CD; and, having drawn BF, to the ſame apply HI = HG, and parallel thereto draw BK, meeting EF in K; then from the center K, with the radius BK, let a circle be deſcribed, *and the thing is done.*

DEMONSTRATION.

Join K, A, and draw KL perpendicular to CD: Then, becauſe of the parallel lines, HG : HI :: KL : KB (*by* 21. 4.); whence, as HG and HI are equal, KL and KB are equal likewiſe: But it is evident, from the Conſtruction, that KA is = KB; therefore KB = KL = KA. *Q. E. D.*

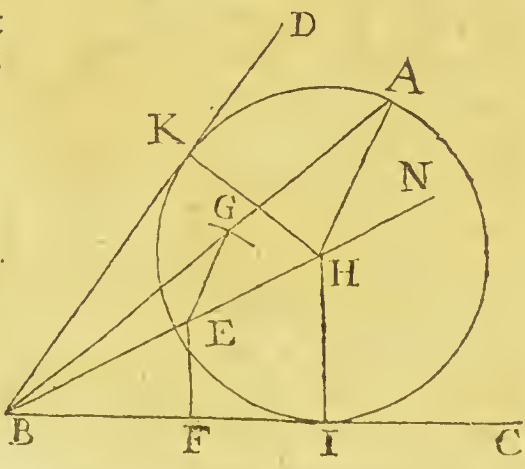
Becauſe two equal lines HI, H*i* may be applied from H to BF (except, when DIH is a right angle) the Problem will therefore admit of two ſolutions. But when a perpendicular let fall from H upon BF is greater than HG; the Problem will be impoſſible. And the like is to be underſtood in the conſtruction of the ſubſequent Problems.

PROBLEM XLIII.

To deſcribe the circumference of a circle through a given point A, *which ſhall touch two right lines* BC, BD *given by poſition.*

CONSTRUCTION.

From the point of concourſe B of the two given lines, draw BA; and alſo BN, to biſect the angle CBD; from any point E in BN, upon BC, let fall the perpendicular EF, and to BA apply EG =



EF,

EF, parallel to which draw AH, meeting BN in H; then from the center H, with the interval HA, let a circle be deſcribed, *and the thing is done.*

DEMONSTRATION.

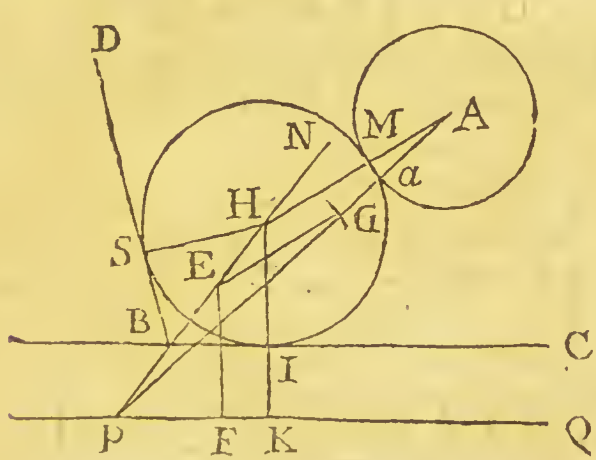
Upon BC and BD let fall the perpendiculars HI and HK; which are manifeſtly equal, becauſe (*by Conſtr.*) the angle HBI = HBK: Moreover, as EF and EG are equal, HI and HA are alſo equal (*by* 21. 4.). *Q. E. D.*

PROBLEM XLIV.

To deſcribe a circle, which ſhall touch a given circle AaM, *and two right-lines* BC, BD, *given by poſition.*

CONSTRUCTION.

Draw PQ parallel to BC, at the diſtance of the radius A*a*; and through the point of concourſe B of the two given lines, draw NBP, biſecting the angle CBD, and meeting PQ in P; moreover, from any point E in PN, upon PQ, let fall the



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perpendicular EF, and from the same point, to PA, apply EG = EF; draw AH parallel to EG, meeting the periphery of the given circle in M, and the right-line PN in H, from which last point, as a center, through M, describe the circumference of a circle; *and the thing is done.*

DEMONSTRATION.

Draw HS perpendicular to BD, and HK to PQ, intersecting BC in I.

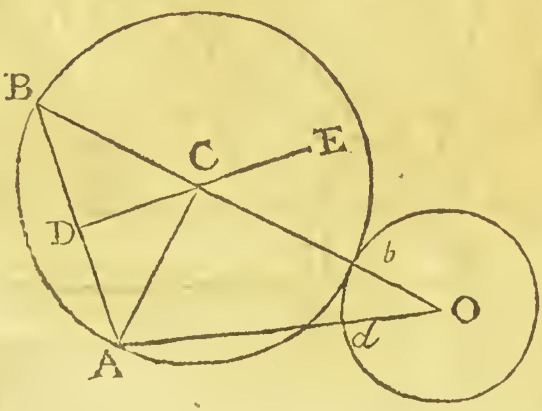
Then, because EG and EF are equal (*by Constr.*), HA and HK (*by* 21. 4.) are likewise equal; from which take away KI = AM (A*a*), and the remainders HI, HM will be equal: But, it is evident, that HI is = HS, because BH bisects the angle IBS; therefore HI = HM = HS. *Q. E. D.*

PROBLEM XLV.

To describe the circumference of a circle through two given points A, B, *and which shall also touch another circle* Odb, *given in position and magnitude.*

CONSTRUCTION.

Bisect the given distance AB with the perpendicular DE, in which (*by Prob.* 15.) let the point C



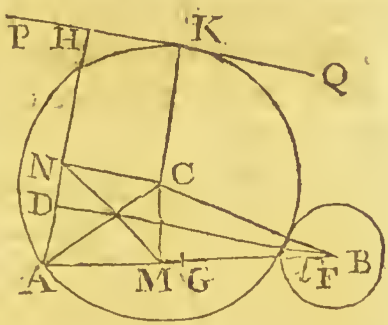
be

be ſo determined, that CO (when drawn) ſhall exceed CA by the radius O*d* of the given circle. Then that point, it is manifeſt, will be the center of the circle to be deſcribed.

PROBLEM XLVI.

Through a given point A, *to deſcribe the circumference of a circle, which ſhall touch a given circle* B, *and alſo a right-line* PQ *given by poſition.*

CONSTRUCTION.



Make AH perpendicular to PQ, and BD to AH; and having drawn AB, in it take BF a third-proportional to AB and the radius B*d*, and let AF be biſected in G: then draw MN (*by Prob.* 35.) ſo that MN, NH, and MG may be in the ſame given ratio, among themſelves, as BD, BA, and B*d*: and at M and N let two perpendiculars be erected on AB and AG; which will interſect each other in the center C of the required circle.

DEMONSTRATION.

Let CA and CB be drawn, and alſo CK perpendicular to PQ. Becauſe MN : NH : : BD : BA (*by Conſtr.*) : : MN : AC (by 22. 4.), thence is NH (CK) = AC. And ſince (*by Conſtr.*) NH (AC) : MG : : BA : B*d*, it is alſo manifeſt, from the Lemma on p. 222. that BC = AC + B*d*. *Q. E. D.*

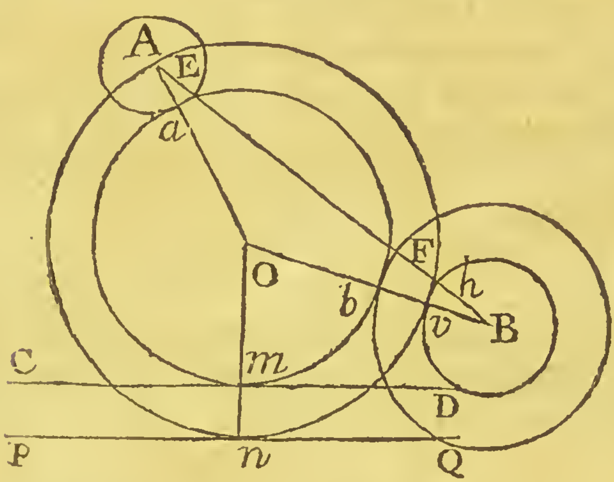
PRO-

PROBLEM XLVII.

To describe a circle O*amb*, *which shall touch two given circles* AE*a*, BF*b*, *and also a right-line* CD, *given by position.*

CONSTRUCTION.

From the radius BF of the greater circle, take away F*b* equal to the radius AE of the lesser; and from the center B, with the interval B*b*, describe the circle B*bv*; also draw PQ parallel to CD, at the distance of F*b* or AE: Then, by the



last Problem, let the center O of a circle be found, whose circumference shall pass thro' A, and touch PQ and B*bv*; and the same point O will, likewise, be the center of the required circle O*amb*.

DEMONSTRATION.

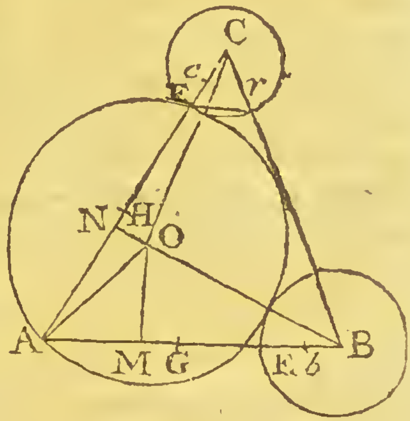
Draw O*n*, perpendicular to PQ, cutting CD in *m*; also let OA and OB be drawn intersecting the circles

circles A and B in a and v: Then, since (*by Constr.*) $AO = vO = nO$, and $Aa\,(AE) = bv\,(Fb) = nm$, let these last be respectively taken from the former, and there will remain $Oa = Ob = Om$. *Q. E. D.*

PROBLEM XLVIII.

To describe the circumference of a circle through a given point A, *which shall touch two other circles* B *and* C, *given in position and magnitude.*

CONSTRUCTION.



To the centers of the given circles, draw AB and AC; in which take Bb = a third proportional to AB and the radius BE; and Cc = a third-proportional to AC and the radius CF: bisect Ab and Ac in G and H, and let Fr be drawn parallel to AB, meeting BC in r. Then (*by Prob.* 34.) draw AO, so that OM and ON being drawn perpendicular to AB and AC, the three lines AO, GM and HN shall have the same given ratio among themselves, as AB, BE and Fr. Then shall the point O be the center of the required circle: For, since (*by Constr.*) $AO : GM :: AB : BE$; and $AO : HN\ (:: AB : Fr) :: AC : CF$; it is manifest, from the Lemma on p. 222. *that* $OB = AO + BE$; and $OC = AO + CF$. *Q. E. D.*

In

In the very ſame manner, a circle may be deſcribed to touch three given circles; the Problem amounting to no more than, *To find a point from whence lines, drawn to three given points, ſhall have given differences:* Since a point ſo found, will always be the center of the required circle, as well when the three given circles are to touch that circle *inwardly*, as when they are all required to touch it *outwardly.*

The End of the GEOMETRICAL CONSTRUCTIONS.

NOTES

GEOMETRICAL and CRITICAL

ON THE

Elementary Part of this WORK.

AXIOM 10. *Book* I. What is here laid down, as an Axiom, would, more properly, have been made a propoſition, had it admitted of ſuch a demonſtration, as is perfectly conſiſtent with geometrical ſtrictneſs and purity. But the laying of one figure upon another, whatever evidence it may afford, is a mechanical conſideration, and depends on no *Poſtulate.*

Theor. 4. *and* 5. *Book* I. There is ſcarce any thing more obvious to ſenſe, and at the ſame time more difficult to demonſtate, than the firſt, and moſt ſimple properties of parallel lines. Even when we have (in Theor. 4.) proved the poſſibility of the exiſtence of ſuch lines, we cannot from thence infer, that their diſtance from one another is every where the ſame; without having recourſe to an Axiom, which, though very evident to ſenſe, cannot be demonſtrated. Theſe difficulties wholly ariſe from our not having any properties, previouſly demonſtrated, whereby the progreſs of a right-line, produced out, can be traced, with reſpect to its diſtance from ſome other right-line aſſigned;

aſſigned; nothing leſs being required here, than the proportionality of the ſides of equiangular triangles; which is not proved before the middle of the Fourth Book, and which depends upon theſe very principles.

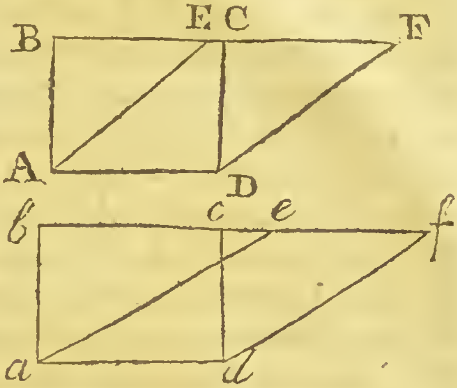
Schol. to Theor 5. *Book.* I. As there are ſeveral conditions requiſite to make up the definitions of a rectangle and ſquare, it was neceſſary to ſhew here, that the ſeveral properties aſcribed to thoſe figures, are not incompatible one with another. *Euclid* is very ſtrict in this particular, and never undertakes to demonſtrate any thing relating to a figure, till he has proved the poſſibility of the exiſtence of ſuch figure by an actual conſtruction.

Theor. 22. *Book* I. This propoſition, which is not in *Euclid*, is of conſiderable uſe, being often wanted in determining the *Maxima* and *Minima*, in mathematical enquiries.

Theor. 27. *Book* I. This Theorem, though not in *Euclid*, is alſo very uſeful, at leaſt, to our deſign: by it we are not only enabled to divide a right-line into any number of equal parts, without the help of proportions, but alſo to demonſtrate that moſt important propoſition, *That the homologous ſides of equiangular triangles are proportional.* It is true, the method purſued here, is not exactly conformable to the idea of proportions delivered in the 6th Def. of *Euclid*'s 5th Book. But, even in that light, the demonſtration will be equally eaſy, without inferring it from the proportionality of triangular ſpaces.

Theor.

Theor. 1. *Book* II. This propoſition, which is not mentioned by *Euclid*, may be thought unneceſſary; but it muſt either be demonſtrated, or aſſumed, as it is wanted in almoſt every propoſition of the ſecond book: By means of it, we alſo arrive at a general, and very eaſy demonſtration of that important Theorem, *That all parallelograms* (AEFD, aefd) *which ſtand upon equal baſes, and have equal altitudes, are themſelves equal*: For, when it is known that theſe parallelograms AEFD, *aefd*, are equal, reſpectively, to rectangles ABCD, *abcd*, of the ſame baſe and altitude (which is proved in Prop. 2.) it is alſo manifeſt that they muſt be equal to one another, as their equal rectangles ABCD, *abcd* are ſhewn *(by Theor.* 1.*)* to be equal, the one to the other. This determination is more general than that given by *Euclid*, in the 36th Prop. of his firſt book; where he demonſtrates the equality of parallelograms, whoſe equal baſes are in the ſame ſtraight line: Which may, perhaps, be thought ſufficient for the whole; becauſe, if the baſes are not in the ſame right-line, one of the two figures may be conceived to be removed, and ſo placed, that its baſe ſhall be in the ſame right-line with the baſe of the other.—But, that theſe were not *Euclid*'s ſentiments, is evident from this; He hints at no ſuch thing: And had he approved of this ſort of demonſtration, his 36th Prop. would have been intirely uſeleſs; as nothing more (after the 35th) would be neceſſary, in order to a general demonſtration, than barely to place one baſe upon the



S other.

other. But it is certain, that this is a kind of demonſtration, which *Euclid* never has recourſe to, when the thing in hand can be done without it. For which reaſon I cannot help wondering a little, that that very accurate Geometer, *Profeſſor Simpſon* of *Glaſgow*, ſhould make uſe of *it*, where (I apprehend) *Euclid* would not. The place, I have now particularly in my eye, is the addition (for it cannot properly be called a corollary) made by him to the firſt propoſition of the ſixth book: Which addition would have been quite unneceſſary, had what is above remarked, reſpecting the equality of parallelograms, been fully eſtabliſhed in the ſecond book: For then the demonſtration, that parallelograms, having equal altitudes, are as their baſes, would nothing differ from that whereby parallelograms, ſtanding between the ſame parallels, are proved to be in proportion as their baſes.

Theor. 9, 11, 12, 13. *Book* II. Theſe four Theorems, tho' not in *Euclid*, are of very conſiderable uſe, particularly the two firſt of them.

Theor. 1. *Book* III. This eaſy propoſition is added in order to give the learner a proper, and more preciſe idea of the quantity of an angle, and of its diviſion in practical uſes.

Theor. 2, 3, 4, 5, 6, 7, 8. *Book* III. Theſe ſeven propoſitions comprehend all that is moſt material in the firſt ſeventeen Theorems of *Euclid*'s 3d book.—As there is no where, in this author, ſo long a run of propoſitions together that are leſs entertaining to learners, or of leſs real importance, than the greater part of the ſaid Theorems; I thought it would be of uſe to reduce the ſubſtance of them into a leſs compaſs: And I flatter myſelf, that I have not ſucceeded ill, in this particular.

Theor.

Theor. 24, 25, 26, 27, 28. *Book* III. These Theorems, which are all of very confiderable ufe, will not, I flatter myfelf, appear lefs plain by being proved independent of proportions, as the demonftrations here given are, not only more concife, but depend alfo upon fewer principles.

Def. 12. *and* 13. *Book* IV. The explication here given, is not ftrictly conformable to the idea of ancient *Commentators*, but is delivered in a fenfe fomewhat more general. With them, *the compofition and divifion of ratios*, extends to thofe cafes only, where the fum, or difference of each antecedent and its confequent, is compared with the confequent. When the antecedent is compared with its excefs above the confequent, this they call the *converfion of ratios*. But in fuch cafes where the antecedent is lefs than the confequent, and where the fum, or difference of the antecedent and confequent is required to be compared with the antecedent; it does not appear that any *terms* have been given to fignify fuch a comparifon. *Profeffor Simpfon* thinks, that the definitions we have, are not *Euclid*'s, but an addition by *Theon*; which to me appears highly probable: This at leaft feems clear, that thefe definitions ought, either to have been extended to a greater number, or elfe to have been rendered more general. For this reafon I have, after the example of modern Geometers, given the fignification of thofe terms, fo as to include all the feveral cafes: And this, I thought, might be done with the greater propriety, as the truth of whatever is here underftood, depends upon the fame demonftration.

Theor. 20, 21, 22, 23. *Book* IV. All thefe Theorems, tho' not in *Euclid*, are of confiderable ufe:

by the two former, the demonſtration of ſeveral others is rendered more eaſy; and the latter have been applied, to good purpoſe, in the analytical determination of ſome difficult geometrical problems.

Prob. 3, 4, 5, 6. *Book* V. In the demonſtrations here given, it may be thought, that I have affected an unneceſſary exactneſs, by making them depend on *Axioms* alone. But I was willing that theſe fundamental propoſitions ſhould have the ſame foundation and evidence, as they appear to ſtand upon in *Euclid*; without referring to any thing derived in virtue of the 4th Poſtulate. But, whether I judged well, or ill, in this particular, is of little conſequence, as the demonſtrations here given, are not leſs plain, and but very little longer, than they otherwiſe would have been.—The Problems themſelves might, indeed, have been given along with the Theorems, as they became neceſſary, according to the method purſued by *Euclid*; whereby any objection, of this ſort, might have been avoided. But, beſides ſome ſmall convenience to the learner, there is a real advantage in having the Problems all together, after the Theorems; ſince, from a great choice of properties, ready demonſtrated to our hands, we are often able to arrive at a ſhorter and better conſtruction, than could poſſibly be given from ſuch properties alone as are antecedent to *Euclid*'s ſolution of the ſame Problem; his method of writing having obliged him to introduce the leading Problems as ſoon as poſſible, in order to evince and eſtabliſh the conſiſtence of his definitions, and to open his way in a regular manner to the many uſeful Theorems thereon depending. And it is for this reaſon alone, that many of his Conſtructions are not ſo well adapted to practice, as thoſe in common uſe.

Upon

Upon which account, ſome have been precipitate enough to blame him; not ſeeing, or conſidering, that ſuch conſtructions, tho' not ſuited to anſwer every purpoſe, were the moſt proper for his plan, and the beſt that could be given in the places where they ſtand.

Prob. 8. *Book* V. The reaſoning in this propoſition, to prove that the two circles will cut each other, may, to ſome, appear needleſs. *Profeſſor Simſon* (at p. 359. of his *Euclid*, 4to Edit. 1756) has been a little ſevere upon me, on this head, for attempting to ſupply, what I thought, a ſmall defect in *Euclid*. "Who is ſo dull (ſays he) tho' only beginning to "learn the Elements, as not to perceive that the "circle deſcribed from the center F, &c." It is not without a real concern that I here ſee this able Geometer drop his own character ſo far, as to expreſs himſelf in a manner ſo very *ungeometrical*. If the thing is, indeed, eaſy to be perceived, it muſt be ſo, either, as an immediate object of the ſenſes, that is, in plain terms, by inſpection; or elſe it muſt be in conſequence of geometrical reaſonings antecedent to the thing itſelf. Now I am clear that he would not be thought to mean the former; and, as to the latter, nothing had been given from whence the evidence of the inference could be ſo clearly ſeen: For, tho' it is proved, that any two ſides of every triangle are greater than the third ſide, it would be abſurd to urge that concluſion in the caſe before us; becauſe the queſtion here, is, whether a triangle, under certain ſpecified conditions, can, or cannot be formed? and, therefore, to conclude any thing from the properties of triangles, would be ridiculous, and nothing leſs than begging the queſtion. That the determination propoſed limits the Problem, no body will diſpute: But then, is it not neceſſary that

this ſhould be proved, in Elements of Geometry, where a reaſon for every thing is, or ought to be given? From this conſideration, I cannot intirely approve of the emendation propoſed by this Editor to *Euclid*'s 24th Prop. Book I. For, tho' the addition there made, does indeed reſtrain the propoſition to one caſe; yet this ought to have been demonſtrated, by ſhewing that the point F (vid. p 29.) muſt in conſequence of ſuch reſtriction, neceſſarily fall below the line EG; but this is not done. Many other inſtances might be produced to ſhew, that this gentleman, who often appears a little too haſty and ſevere in his cenſures, is not, himſelf, every where equally guarded. In Prop. I. Book III. he bids you to draw a ſtraight-line within a circle, without ſpecifying that it muſt terminate in the circumference; and, what is a great deal worſe, he here very improperly uſes the word *within*; when the propoſition itſelf is laid down in order to prove, in the ſubſequent one, that ſuch line muſt neceſſarily fall *within* the circle. Theſe are, it is true, but little matters; but leſs than theſe have fallen under this gentleman's notice. At p. 358, M. *Clairaut* is glanced at, for an inadvertency of this ſort. And, in the note at the head of p. 415. it is ſaid, "The words, *for a ſtraight-line cannot* "*meet a ſtraight-line in more than one point*, are left "out, as an addition by ſome unſkilful hand; for "this is to be demonſtrated, not aſſumed." Now can it poſſibly ſhew any want of ſkill in an editor, to refer to an axiom which *Euclid* himſelf had laid down (Book I. N°. 14.) and not to have demonſtrated, what no man can demonſtrate?

Prob. 16, 17, 18. *Book* V. Theſe three Problems, tho' not ſo frequently wanted as ſome of the preceding ones, are nevertheleſs of very conſiderable uſe. The two laſt of them are the ſame, in

in effect, with the 28th and 29th of *Euclid's* 6th book; but are here put down in a manner rather more commodious.

Axiom, p. 131. This Axiom is substituted instead of the common definition of equal solids, which, I really think, is too bad to be the work of *Euclid*. "It is not a definition, but a proposition, "whose truth or falshood ought (as a very judi-"cious writer observes) to be demonstrated, not "assumed." Neither is it at all conformable to *Euclid's* manner of writing, where he establishes the foundation whereon the equality of plane figures is grounded; which he does, not by means of a definition, but from the application of two of the most simple figures to each other; so that, from the coincidence of their bounds, their equality may appear manifest. And this method we have pursued in treating of solids; without which a clear and distinct idea of their equality can be no more obtained, than of the equality of plane figures independent of the 4th Prop. of the first book, which is our 10th Axiom. I should have said a good deal more on this head, but I find that *Professor Simson* has already placed this matter in so strong and clear a light, as to render any farther apology, or comment, unnecessary here. Tho' I must confess, that, had this gentleman's work come into my hands * before the elementary part of my own had

* *This did not happen till the middle of November* 1759; *when being in town, in company with my bookseller, and being pressed by him to finish; I acquainted him that every thing was actually ready except the Preface, which would cost me some pains, since it would be necessary to obviate some objections, particularly with regard to the reasons for my rejecting* Euclid's *definition of equal solids, and building upon a different foundation. On which, he immediately let me know, that Mr.* Rob. Simson *had already cleared*

had been intirely printed off, the definition of similar solids, which I have given, from *Euclid*, would have been delivered under a form somewhat different. For, tho' it involves no absurdity, as it now stands, yet there are certain cases (but such indeed as do not occur in any Elements of Geometry) where it will not afford the precise idea it ought to convey.

Postulatum, p. 131. The unsatisfactory and inconclusive demonstration given to *Euclid*'s 2d Prop. Book XI. (by *Euclid* himself, or some less skilful *Editor*) seemed to render something of this sort necessary. In that demonstration it is taken for granted, that one part of the triangle, at least, must be in the same plane. But it has been very justly observed, that a curve surface may be bounded by three right lines: Nor does it seem easy to form a clear idea, that even a part of any one of the three lines will be in the same plane with one of the others, unless by conceiving a plane to be turned about upon the one, till it meets with, or falls upon some point in the other. And I have the satisfaction in this particular, to see my sentiments exactly agree with those of a very good judge, whose name, I have, more than once, had occasion to mention in these notes. It is true, he makes that a Theorem, which I lay down as a Postulate. But, since a plane can no more be turned about upon a line, than a line can be drawn from one point to another, it seemed to me, that the one

up that point; and expressed his surprize that a work of so much repute, wherein (he told me) my own name was more than once mentioned, had not come into my hands. A copy of which I received from him the next morning. In consequence whereof I changed my first design of writing a long Preface; thinking it would be better to give what I had to offer, in notes, after the example of this Editor.

was

was as properly a Poſtulate, as the other. However, whether this be, or be not allowed, is immaterial, as the degree of evidence is preciſely the ſame. There is, indeed, one reaſon, why this Theorem, or Poſtulate, ought to have preceded that gentleman's demonſtration of Prop. 1. Book XI; It is there wanted: For, in the Corollary, on which that demonſtration is made to depend, the lines AB, BD, BC are ſuppoſed to be all in the ſame plane; which ought by no means to be aſſumed in the firſt of the 11th. *Euclid*'s 10th Axiom, which that Corollary is intended to ſupply, and by which the propoſition is uſually demonſtrated, is not limited by any ſuch reſtriction.

Theor. 12. *Book* VII. This propoſition is added on account of its uſe, being the foundation on which the whole *art of perſpective* in a manner depends.

Theor 25. *Book* VII. From this Theorem, which is very extenſive in its application, ſeveral others of conſiderable note may be deduced: one, or two of which, for the ſake of the learner, I ſhall here derive, and put down by way of example.

Let A, B, C, D denote four lines in continual proportion.

$$\text{Then, ſince} \left\{ \begin{array}{l} A : B :: C : D, \\ A : B :: B : C, \\ A : B :: A : B, \end{array} \right\} \text{it follows } (\textit{from Theor. } 25.) \text{ that}$$

$$A^3 : B^3 :: CBA : CBD :: A : D \ (\textit{by } 22.$$

7.) or that, of four lines in continual proportion, the cube of the firſt is to the cube of the ſecond, as the firſt line is to the fourth.

Again, let, $A : a :: B : b :: C : c$ (where A, *a*; B, *b*; C, *c*, may be ſuppoſed to repreſent the homologous ſides of two ſimilar parallelepipedons).

Then,

Then, ſince $\left\{\begin{array}{l} A : a :: B : b \\ A : a :: C : c \\ A : a :: A : a \end{array}\right\}$ it alſo follows,

that $A^3 : a^3 :: BCA : bca$; or, that ſimilar parallelepipedons are to one another, as the cubes of their homologous ſides.—The proportionality of ſimilar parallelepipedons, deſcribed upon proportional lines, is alſo included in the ſame Theorem; being no other than that caſe of it, where the propoſed ratios are all equal.

Theor 5. *Book* VIII. The demonſtration of this Theorem might have been delivered under a form ſomewhat different, by aſſuming two other ſolids (without regard to figure) the one leſs, and the other greater than the propoſed parallelepipedon IP, and proving that cylinder muſt, *alſo*, be greater than the one, and leſs than the other: which is done by means of *Lem.* 1. that is, by taking P*m*, or P*t*, ſo ſmall a part of IP, as to be leſs than the difference between the given parallelepipedon and either of the ſaid ſolids: from whence the demonſtration will proceed on, in the ſame manner in which we have given it. But, as theſe additional conſiderations would have increaſed the number of ſchemes, and lengthened the proceſs, without adding one jot to the degree of evidence, it was thought proper, for the ſake of the learner, to omit them.

Theor. 8. *Book* VIII. This Theorem is not ſo uſeful as the Corollaries that follow from it, which are all of very great importance: In the 3d and 4th of them, the proportion of all kinds of priſms and pyramids is aſſigned, without the aſſiſtance of the uſual demonſtrations given for this purpoſe; which, tho' ſufficiently evident in themſelves, are often found a little perplexing to learners, on account

count of the ſchemes, wherein ſo great a number of lines is neceſſary.

Nothing, in the courſe of theſe notes, has been ſaid relative to the Theorems on proportions, tho' ſo nice and critical a ſubject, and tho' the method I have therein purſued may ſtand in need of ſome apology. But, indeed, the whole of what I have to offer on this head, was too much to be comprized in the compaſs of one ſingle note, and could not ſo properly be delivered in ſeveral detached ones: For which reaſon, I ſhall here throw together all that I have to advance on that ſubject.

There are two objections that may be brought againſt the method in which proportions are treated of in this work; the one, grounded on the impoſſibility of dividing every magnitude into equal parts; and the other, on the incommenſurability of two or more magnitudes of the ſame kind, when compared with each other. The firſt of theſe objections appears, to me, to have very little weight. For, tho' a magnitude may be ſo conſtituted, that the diviſion of it into an aſſigned number of equal parts cannot be, *actually*, effected by any geometrical conſtruction; yet it is no leſs evident, for that reaſon, that every ſuch magnitude has really its third, fourth, or other aſſigned part, tho' we are at a loſs how to take it; or, in other words, it ſeems very clear to conceive, that in every propoſed magnitude, whatever its figure may be, a leſs magnitude is contained, which, repeated an aſſigned number of times, ſhall be equal to the magnitude given. If, as the moſt rigid judges allow, every plane figure is equal to *ſome* ſquare, and every ſolid equal to *ſome* parallelepipedon; then the parts of the ſquare, or parallelepipedon, which are actually determinable by a geometrical

metrical conſtruction, will alſo be like parts of the figure firſt propoſed, and ſuch as we conceive to be taken.

The other objection, depending on the incommenſurability of magnitudes, is a matter of real difficulty; which we have taken ſome pains to obviate, in the *Scholia* to our 3d and 7th Theorems. *Euclid*, himſelf, ſeems to have been not a little embarraſſed with *it*, if we may be allowed to judge from the different methods he has left us in his 5th and 7th books; the former whereof, which is ſuited to include the buſineſs of incommenſurables, being nothing near ſo eaſy and natural as the latter: It has, it is true, the advantage of being general; but, that the principles whereon it is grounded, are neither ſo ſimple, nor ſo evident as might be wiſhed for, the many diſputes about them, ſince *Euclid*'s time, by Geometers of the firſt rank, will in a great meaſure evince. And farther, it ſeems ſufficiently plain, from *Euclid*'s own authority, that he himſelf was not intirely pleaſed with his own performance on this head; or that he was convinced (at leaſt) that it had not *every advantage*: For, otherwiſe, it will be very difficult to account for his having demonſtrated many things in his 7th book, by another method, whoſe demonſtrations had been actually given before, in the 5th, under a different form. For theſe reaſons, when I ſee the extravagant commendations that have been laviſhed on this 5th book of *Euclid*, I am no farther convinced by them, than that great men may ſometimes launch out too far in behalf of opinions which they have adopted. And I believe that, whoever has read the notes on the 5th book, by that great reſtorer of *Euclid*, *Profeſſor Simſon*, will be apt to conclude, that thoſe high encomiums are a little miſapplied. Indeed, if all that is advanced in thoſe notes be allowed of, I think the author of them has proved too

too much; and this ſuperb fabric of proportions, reared with ſo much art, ſtands upon a tottering foundation. It is not by choice that I go out of my way to play the *critic*; but as the writers againſt the vulgar and indiſtinct notion of proportions (as they term it) are very ſevere in their cenſures, and aſſume a great ſuperiority, from the boaſted *accuracy of their reaſonings*, it may be neceſſary to ſhew my reader, that, tho' what he is *here* taught on proportions, is liable to ſome objections, the method which ſome ſo greatly prefer, has *alſo* its difficulties; and that there are *other objections* to it beſides its *obſcurity*. And this I ſhall make appear from this learned Commentator's own authority and conceſſions; and in order thereto, ſhall firſt refer to his note on Prop. 10. which proceeds thus. "It was "neceſſary to give another demonſtration of this "propoſition, becauſe that which is in the *Greek*, "and *Latin*, or in other editions, is not legitimate. "For the words *greater*, *the ſame*, or *equal*, *leſſer*, "have a quite different meaning when applied to "magnitudes and ratios, as is plain from the 5th "and 7th definitions of Book 5. by help of theſe let "us examine the demonſtration of the 10th Prop. "which proceeds thus, &c." He then goes on, in a long note, to ſhew the inſufficiency of a demonſtration, which had been received, by all, as perfectly genuine and ſatisfactory; and at laſt comes to this concluſion. "Wherefore the 10th Propo-"ſition is not ſufficiently demonſtrated. And it "ſeems, that he who has given the demonſtration "of the 10th Propoſition, as we now have it, in-"ſtead of that which *Euclid* or *Eudoxus* had given, "has been deceived in applying what is manifeſt, "when underſtood of magnitudes, unto ratios, "*viz.* that a magnitude cannot be both greater "and leſs than another. That thoſe things which "are equal to the ſame are equal to one another,

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"is a moſt evident Axiom when underſtood of "magnitudes, yet *Euclid* does not make uſe of it "to infer that theſe ratios which are the ſame to "the ſame ratio, are the ſame to one another; "but explicitly demonſtrates this in Prop. 11. of "Book 5. The demonſtration we have given of the "10th Prop. is no doubt the ſame with that of "*Eudoxus* or *Euclid*, as it is immediately and di-"rectly derived from the definition of a greater "ratio, *viz.* 7th of 5."

Here the weight of the objection reſts on its not having been proved, that, of three given magnitudes A, B, C, the ratio of A to C could not, at the ſame time, be both greater and leſs than that of B to C. But, if in the demonſtration, here rejected as inſufficient, there is any real flaw, it is chargeable on the definition of a greater and leſs ratio, as the reaſoning from it, is clear, ſtrong, and perfectly ſcientific. And I would ſeriouſly aſk the Contemners of *the vulgar and confuſed notion* of proportions, if a definition, by which it cannot be known, whether the ratio of the firſt to the ſecond of four given magnitudes, may not, at the ſame time, be both greater and leſs, than that of the third to the fourth, is really calculated to afford thoſe very accurate ideas they pretended to? This Commentator has too much penetration not to be aware of the force of this objection, which he has attempted to obviate in one particular caſe. But the new propoſition given by him, for that purpoſe, ought to have preceded the 10th, and to have been demonſtrated, independent of it. This he alſo ſeems apprized of, when he ſays, that "it cannot "be eaſily demonſtrated without the 10th, as he that "tries to do it will find." But, be this as it will, I am not at all clear that his "demonſtration of "the 10th, is the ſame with that of *Eudoxus* or "*Euclid.*" *Euclid*, or (if you pleaſe) *Eudoxus*, does

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never (that I know of) refer to any definition, till it has been proved, either by an actual construction, or by some demonstration previous to that in hand, that such definition involves no absurdity, or conditions that are incompatible one with another. If, therefore, it was conceived possible, that the definition of a greater and less ratio, could involve so great an absurdity, as that, *by it*, the ratio of A to C might at the same time be both greater and less than that of B to C; this point, according to the method prescribed by *Euclid*, ought to have been cleared up, not by means of propositions derived in virtue of that very definition, but by others antecedent thereto, and independent thereupon. And, to me, the 8th Prop. seems the proper place for the doing of this, where it might be easily introduced, either in the Prop. itself, or by way of Corollary. It is there proved, that if, of three magnitudes A, B, G, the first A is greater than the second B, then certain equimultiples of A and B may be taken such, that being compared with some multiple of C, the multiple of A shall be greater, and that of B less than the said multiple of C. Whence, by the definition of a greater ratio, the ratio of A to C is greater than that of B to C. To which might be added—And because A is greater than B, any multiple whatsoever of A must be greater than the same multiple of B; and, consequently, no equimultiples whatsoever, of A and B can possibly be so taken, that the multiple of A shall be equal to, or less than some multiple of C, and that of B greater than the multiple of C: for, if the multiple of B be greater than the multiple of C, the multiple of A, which is greater than that of B, must also be greater than the multiple of C. Wherefore the ratio of A to C cannot (by the definition) be less (as well as greater) than the ratio of B to C.

But, notwithstanding all that has been proved on this head, either here, or by that gentleman him-

self,

ſelf, the ſame objection occurs again in Prop. 13: where it remains in its full force. For, tho' it be allowed, that "there are ſome equimultiples of C "and E, and ſome of D and F ſuch, that the "multiple of C is greater than the multiple of D, "but the multiple of E not greater than the mul-"tiple of F;" yet it is not demonſtrated, nor in any ſort ſhewn, that other equimultiples of thoſe quantities cannot be taken ſuch, that the very contrary ſhall happen.—If the demonſtration of the 10th Prop. has been juſtly rejected by this gentleman himſelf, as inſufficient, becauſe the impoſſibility of a contrary concluſion had not been ſhewn; can it be thought that this 13th Propoſition is, at this day, ſufficiently demonſtrated, where the ſame objection occurs, and that in a much greater latitude? I have a much better opinion of this Editor's diſcernment, than to imagine, that his paſſing this matter over in ſilence, proceeded from his not being aware of the difficulty; but it ſeems to me, that his great diſlike to the *vulgar* idea of proportion (ſo often teſtified in the courſe of his notes) would not permit him to borrow any thing from thence, however evident, and though this objection, that ſtrikes deep at the very root of proportions, might by means thereof be very eaſily removed. I ſay, the very root of proportions is deeply ſtruck at in this objection; becauſe both the alternation and equality of ratios (*ex æquali ſc. diſt.*) are grounded on the ſaid 13th Prop. and which, therefore, till the objection is removed, muſt be allowed to ſtand upon an uncertain foundation.

The principle hinted at above, whereby the difficulty might be obviated, is, that if a magnitude of any kind be given, or propounded, there may (or can) be another magnitude of the ſame kind which ſhall have to it any ratio aſſigned. This aſſumption Mr. *Profeſſor* will by no means admit of

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(tho' *Euclid* himſelf, in Prop. 2. of his 12th book, has uſed it); and, in a long note on Prop. 18. is angry with *Clavius* for having recourſe to it; affirming, "that the demonſtration (given by means thereof) "is of no force;" and that "the thing itſelf can-"not (as far as he can diſcern) be demonſtrated by "the preceding propoſitions, ſo far is it from de-"ſerving to be reckoned an Axiom, as *Clavius*, "after other Commentators, would have it." That the aſſumption cannot be generally demonſtrated by the preceding propoſitions (nor even by all the propoſitions in the Elements) I readily aſſent to: but then, becauſe a thing, exceedingly obvious in its own nature, cannot be demonſtrated; is it therefore leſs proper for an axiom? I ſhould rather take the other ſide of the queſtion, and maintain that nothing ought to be made an Axiom, which can be demonſtrated. But we are not, it ſeems, allowed to have any idea of proportion but what is contained in the 6th and 8th (or, as this Author makes them, the 5th and 7th) definitions of *Euclid*'s 5th Book. And, in his note on the new Prop. marked A, He is again diſpleaſed with *Clavius*, for thinking it ſufficiently evident, *from the nature of proportionals*, that if, of four proportional magnitudes, the firſt antecedent is greater than its conſequent, the ſecond antecedent will alſo be greater than its conſequent. "As if there was "(ſays he) any nature of proportionals antecedent "to that which is to be derived and underſtood "from the definition of them." Now I cannot help thinking, with *Clavius*, that there was a nature, or idea of proportion antecedent to that given in the 6th and 8th definitions of Euclid's 5th book: For, that mankind, long before the time of *Euclid*, had ſome way to ſhew, or expreſs, in what degree one magnitude was greater or leſs than another, cannot be doubted: And this was

the firſt, and natural idea of proportion: And I look upon thoſe definitions, as refinements, only, on the ſimple and natural idea, in orber to take in the buſineſs of incommenſurables; whereby the original notion is ſo much obſcured, that it requires ſome ſkill, even to ſee that it is at all contained in theſe definitions. I intirely agree with this gentleman, that every demonſtration ought to be ſtrictly derived from principles before eſtabliſhed: But then, whether is it more eligible, to have recourſe to an Axiom founded (as all other Axioms are) on the evidence of ſenſe and reaſon, or to an obſcure and perplexed definition, which may, for any thing that has been proved to the contrary, involve an abſurdity?

That there is ſomething very ingenious and ſubtle in the doctrine of proportions, as delivered in *Euclid*'s 5th book, cannot be denied. All that I contend for, is, that the principles on which it is built are *obſcure*, and not ſo firmly eſtabliſhed, as to authorize its partiſans to aſſume that great ſuperiority they lay claim to, in point of geometrical ſtrictneſs.

I have intimated above, that the principle is rejected, by which the conſiſtence of the definition of a greater and leſs ratio might be eſtabliſhed, without much difficulty: But I would not be thought to mean, that the ſame thing cannot poſſibly be effected any other way, becauſe I am ſatisfied that it may be done from the conſideration of multiples alone: But a demonſtration of this ſort is not eaſy.—Were I to treat of proportions from the plan laid down in the 5th book of *Euclid*, I would intirely reject the 10th and 13th propoſitions, and every thing elſe founded on the definition of a greater and leſs ratio, as being of no other uſe in the Elements, than to open the way to thoſe important Theorems on the alternation and equality of ratios; which may be better demonſtrated without them, from the definition of equal ratios alone; which,

which, from the conditions of it, can admit of no abſurdity, and whoſe conſiſtence is evinced in Prop. 15. and ſtill more clearly in the firſt of the ſixth.

In the 14th of the 5th, whereon the alternation of ratios is grounded, it is neceſſary to demonſtrate, " That if, of four proportional magnitudes, of the " ſame kind, A, B, C, D, the firſt be greater than " the third, the ſecond ſhall be greater than the " fourth; and if equal, equal; and if leſs, leſſer." Which may be very eaſily done, independent both of the 10th and 13th, in the manner following.

Firſt, let A be greater than C.

Of A and C (*by Prop.* 8.) let ſuch equimultiples be taken, that the multiple of A ſhall be greater, and that of C leſs, than ſome multiple of B; let E and G be any two ſuch equimultiples of A and C, and F the multiple of B; ſo that E ſhall be greater than F, and G leſs than F; and let H be the ſame multiple of D, as F is of B. Therefore, becauſe E and G are equimultiples of the firſt and third, and F and H alſo equimultiples of the ſecond and fourth; and ſeeing that, (*by Hyp.*) E is greater than F; it is evident, from the definition of equal ratios, that G muſt likewiſe be greater than H: Therefore much more ſhall F (which exceeds G) be greater than H; whence alſo B ſhall be greater than D (*by Ax.* 4.) B and D being like parts of F and H.

When A is leſs than C, it will be demonſtrated in the ſame manner, that B is alſo leſs than D. But when A is equal to C, no new demonſtration is neceſſary; ſince neither the 10th nor the 13th have any thing to do in this caſe.

Again, in the 20th Prop. (in which the 10th and 13th alſo enter) we are to prove, " That if there be

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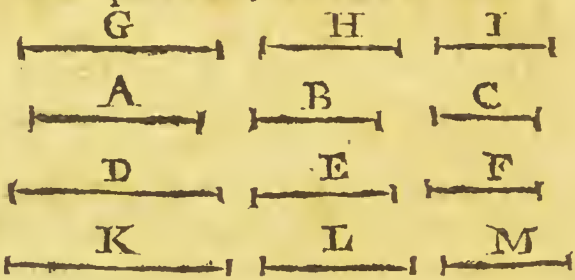
" three magnitudes (A, B, C) and other three (D, " E, F) which taken two and two have the fame ra- " tio (A : B : : D : E, B : C : : E : F) if the firft (A) " be greater than the third (C), the fourth (D) fhall " be greater than the fixth (F); and if equal, equal; " and if lefs, leffer." Which may likewife be done, without the affiftance of either the 10th or the 13th, in the fame manner, above fpecified.

For, if A be greater than C; then of A and C (*by Prop.* 8.) fuch equimultiples may be taken, that the multiple of A fhall be greater, and that of C lefs, than fome multiple of B; let G and I be two fuch equimultiples of A and C, and let H be the multiple of B, fo that G fhall be greater than H, and I lefs



than H; moreover take L the fame multiple of E, as H is of B; and K and M the fame equimultiples of D and F, as G and I are of A and C. Therefore, fince of the four proportionals A, B, D, E, equimultiples G, K of the firft and third, and equimultiples H, L of the fecond and fourth, are here taken, it is manifeft, from the definition of equal ratios, feeing G is greater than H (*by Hyp.*) that K muft alfo be greater than L. And in the very fame manner, becaufe B, C, E, F are proportionals, and H is greater than I, L will likewife be greater than M: Therefore much more fhall K, which exceeds L, be greater than M. And confequently (*by Ax.* 4.) D fhall alfo be greater than F. —When A is lefs than C, the demonftration is the fame: The other cafe, when A is equal to C, does not require, nor indeed admit of any improvement.

F I N I S.